Variational methods for nuclear matter and their applications to neutron stars

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1: Brief review of nuclear equations of state
2: Cluster variational method for core collapse supernovae
3: Variational Method with explicit energy functionals for nuclear matter
4: Applications to the modified Urca process

1: Brief review of nuclear equations of state

Based on

"Nuclear Equation of State for Compact Stars and Supernovae"

G. Fiorella Burgio and Anthea F. Fantina

in

"The Physics and Astrophysics of Neutron Stars" (2018)

Editors: Rezzolla, L., Pizzochero, P., Jones, D.I., Rea, N., Vidana, I.

(arXiv:1804.03020)

Infinite Uniform Nuclear Matter (zero temperature)

Uniform matter composed of nucleons (protons and neutrons)
interacting through nuclear forces

Energy per nucleon is finite: The nuclear force is short-ranged.

$\frac{E}{N}(n_{\rm B}, Y_{\rm p})$ $n_{\rm B}$: Nucleon number density $Y_{\rm p}$: Proton fraction

$$\frac{E}{N}(n_{\rm B}, Y_{\rm p}) = \frac{E}{N}(n_{\rm B}, 1 - Y_{\rm p})$$

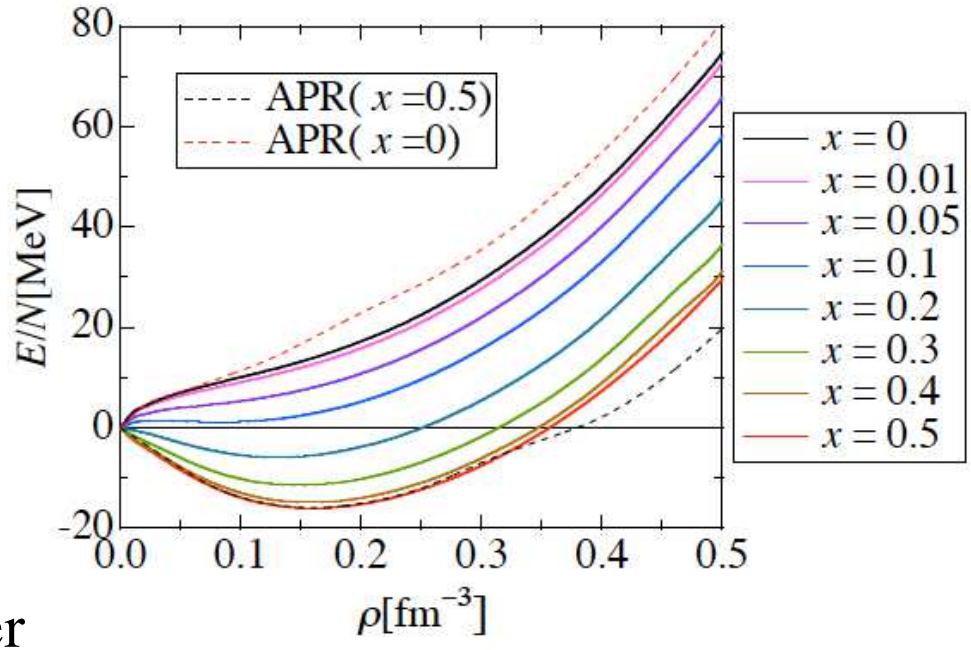
Charge symmetry of the nuclear force

$$0 \le Y_{\rm p} \le \frac{1}{2}$$

$Y_{\rm p} = \frac{1}{2}$ Symmetric nuclear matter

$Y_{\rm p} = 0$ Pure neutron matter

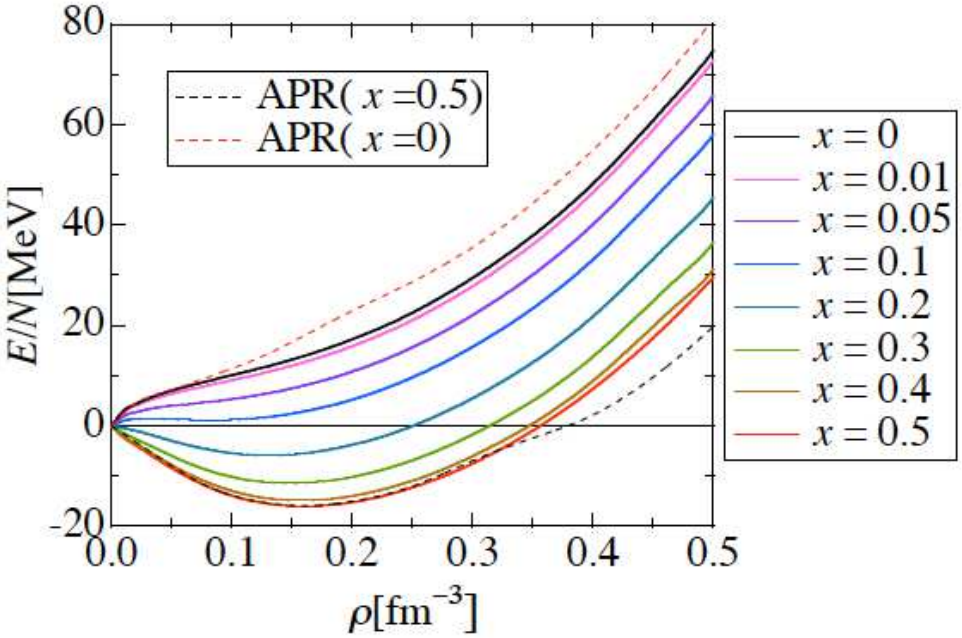
$0 < Y_{\rm p} < \frac{1}{2}$ Asymmetric nuclear matter



Togashi EOS

Experimental Constraints on the Nuclear Equation of State

Parabolic approximation $$\frac{E}{N}(n_{\rm B}, Y_{\rm p}) \sim \frac{E_0}{N}(n_{\rm B}) + \frac{E_{\rm sym}}{N}(n_{\rm B})\,(1-2Y_{\rm p})^2$$



$$\mathrm{Min}\frac{E}{N}(n_{\rm B}, Y_{\rm p}) = \frac{E_0}{N}$$

$$= \frac{E}{N}(n_{\rm B} = n_0, Y_{\rm p} = 1/2)$$

Saturation density

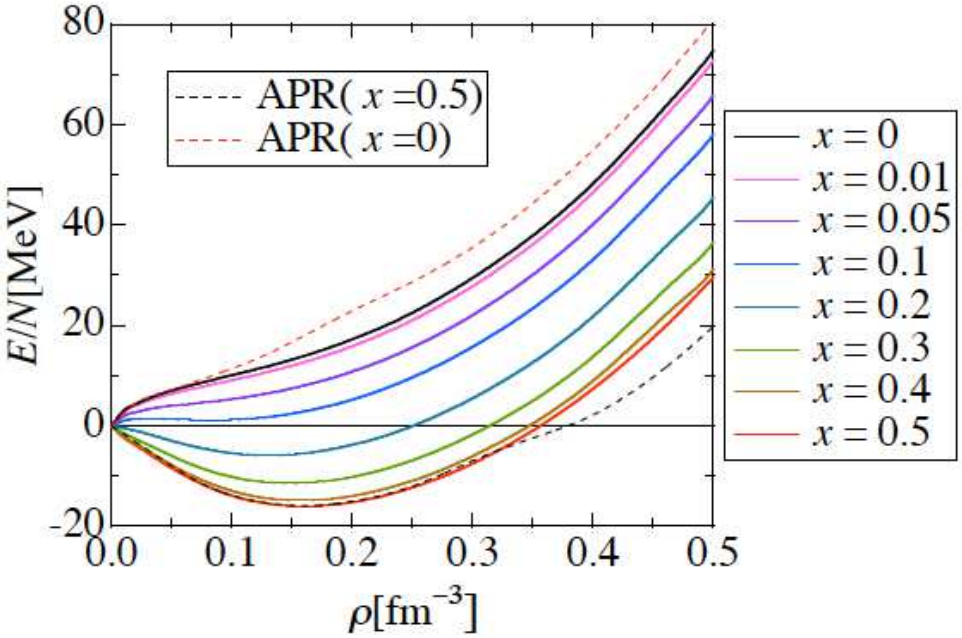
$$n_0 = 0.16\ \mathrm{fm}^{-3}$$

Saturation energy

$$\frac{E_0}{N} = -15.8\ \mathrm{MeV}$$

Experimental Constraints on the Nuclear Equation of State

Parabolic approximation $\frac{E}{N}(n_{\rm B}, Y_{\rm p}) \sim \frac{E_0}{N}(n_{\rm B}) + \frac{E_{\rm sym}}{N}(n_{\rm B})\,(1-2Y_{\rm p})^2$



Symmetry energy

$$\frac{E_{\rm sym}}{N}(n_{\rm B}) = \frac{1}{8}\frac{\partial^2}{\partial Y_{\rm p}^2}\frac{E}{N}(n_{\rm B}, Y_{\rm p})\,|_{Y_{\rm p}=1/2}$$

$$\frac{E_{\rm sym}}{N}(n_0) = 31.6 \pm 2.7 \text{ MeV}$$

Slope Parameter

$$L = 3n_0\frac{\partial}{\partial n_{\rm B}}\frac{E_{\rm sym}}{N}(n_{\rm B}, Y_{\rm p}=1/2)\,|_{n_{\rm B}=n_0}$$

$$= 58.9 \pm 16 \text{ MeV}$$

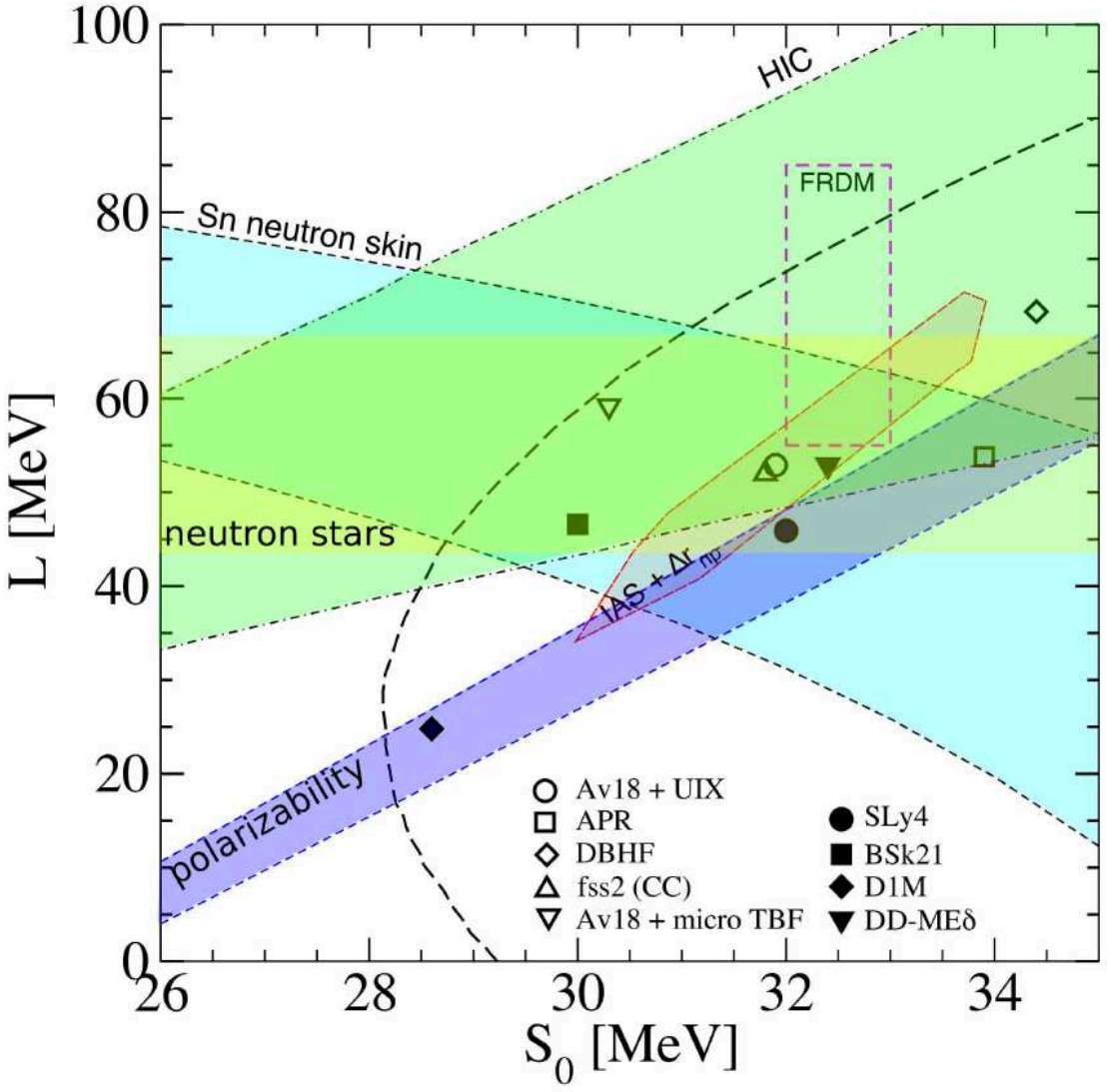
Incompressibility

$$K = 9n_0^2\frac{\partial^2}{\partial n_{\rm B}^2}\frac{E}{N}(n_{\rm B}, Y_{\rm p}=1/2)\,|_{n_{\rm B}=n_0}$$

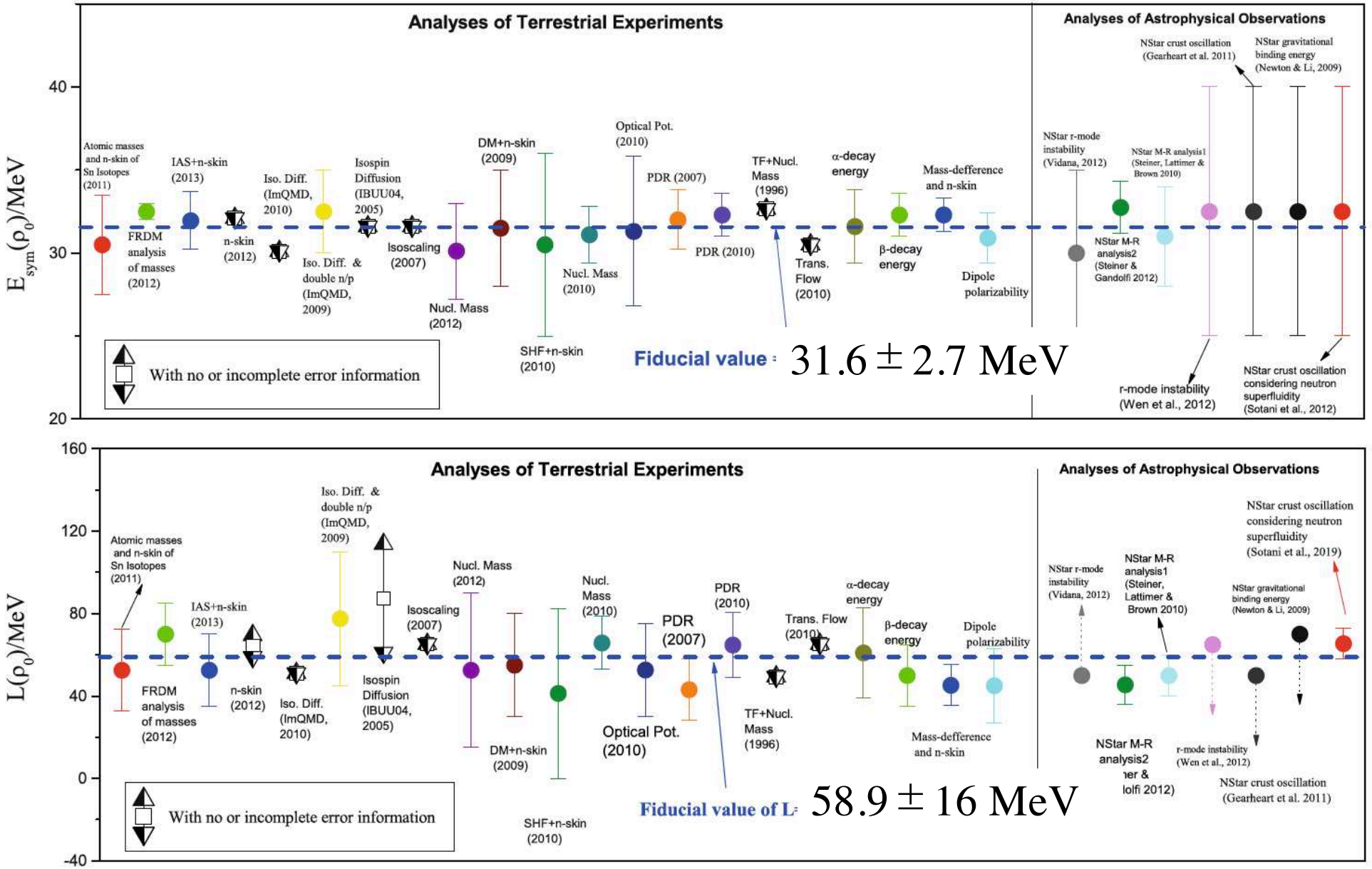
$$= 240 \pm 10 \text{ MeV}$$

Bao- An Li, et al., Eur. Phys. J. A (2019)55: 117.
G. Colo et al., Phys. Rev. C70 (2004) 024307.

Symmetry Energy and Slope Parameter



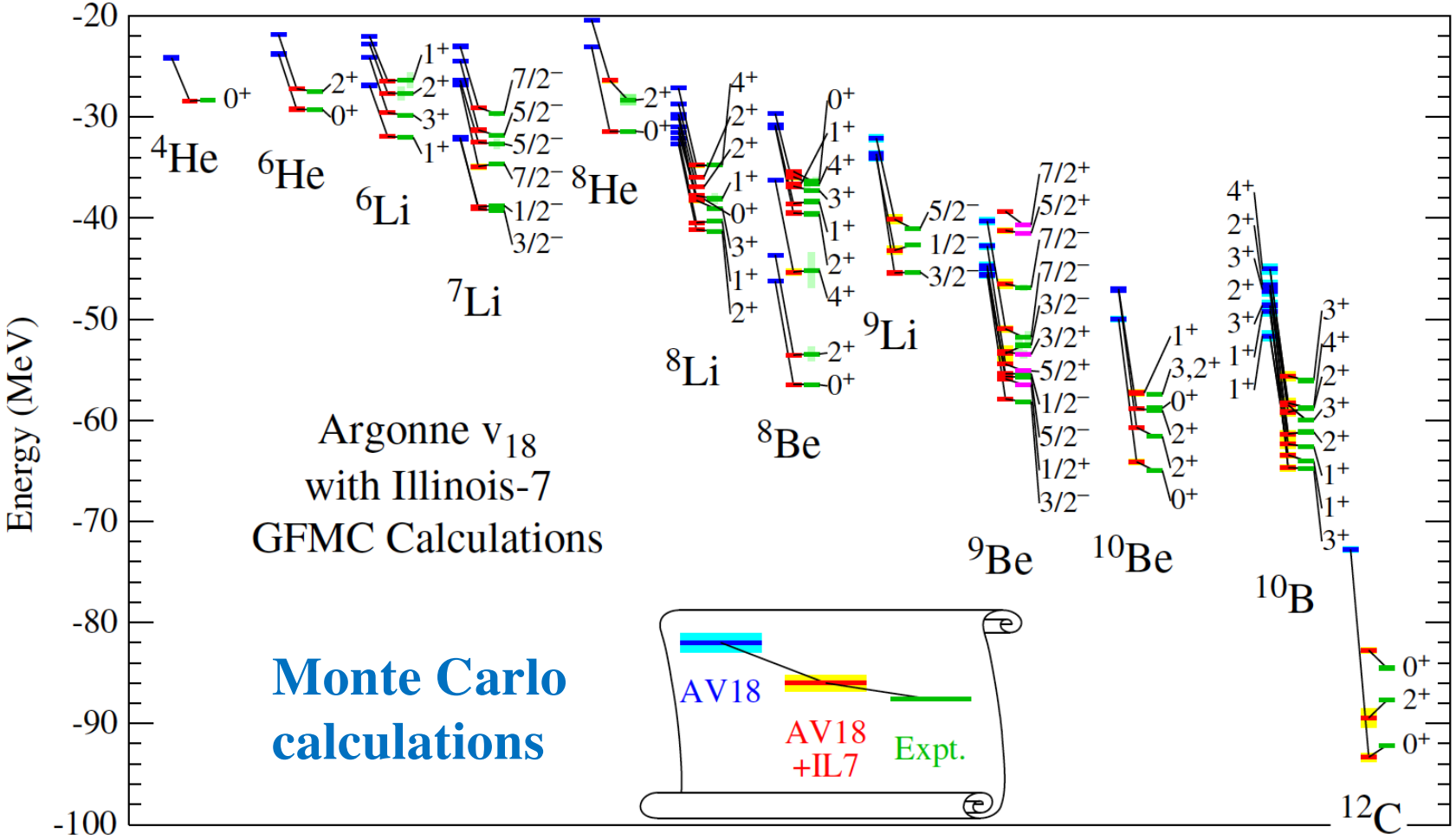
G. Fiorella Burgio and Anthea F. Fantina 2018



Bao- An Li, et al., Eur. Phys. J. A (2019)55: 117.

The Nuclear Hamiltonian (Nonrelativistic)

$$H = -\sum_{i=1}^{N} \frac{\hbar^2}{2m} \nabla_i^2 + \sum_{i<j}^{N} V_{ij} + \sum_{i<j<k}^{N} V_{ijk}$$



J. Carlson et al., Rev. Mod. Phys. 87 (2015) 1067.

Argonne two-nucleon potentials

The AV18 two-body potential

R. B. Wiringa et al., PRC **51** (1995) 38

$$V_{ij} = \sum_{t=0}^{1} \sum_{s=0}^{1} \left[V_{\mathrm{C}ts}(r_{ij}) + V_{\mathrm{T}t}(r_{ij}) S_{\mathrm{T}ij} + V_{\mathrm{SO}t}(r_{ij}) \left(\boldsymbol{s} \cdot \boldsymbol{L}_{ij} \right) \right.$$

Central Tensor Spin-orbit

$$\left. + V_{\mathrm{qL}ts}(r_{ij}) \left| \boldsymbol{L}_{ij} \right|^2 + V_{\mathrm{qSO}t}(r_{ij}) \left(\boldsymbol{s} \cdot \boldsymbol{L}_{ij} \right)^2 \right] P_{tsij} + V_{\mathrm{corr}}$$

Quadratic orbital angular momentum

Quadratic spin-orbit

t: isospin, s: spin

Experimental NN scattering data are well reproduced.

The AV8' two-body potential

$$V_{ij} = \sum_{t=0}^{1} \sum_{s=0}^{1} \left[V_{\mathrm{C}s}(r) + s V_{\mathrm{T}}(r) S_{\mathrm{T}ij} + s V_{\mathrm{SO}}(r) \left(s \cdot L_{ij} \right) \right] P_{tsij}$$

The AV6' two-body potential

$$V_{ij} = \sum_{t=0}^{1} \sum_{s=0}^{1} \left[V_{\mathrm{C}s}(r) + s V_{\mathrm{T}}(r) S_{\mathrm{T}ij} \right] P_{tsij}$$

The Urbana three-nucleon potential

The UIX potential [1]

$$V_{ijk} = V_{ijk}^{R} + V_{ijk}^{2\pi}$$

V_{ijk}^{R} : Repulsive potential

$$V_{ijk}^{\mathrm{R}} = U \sum_{\mathrm{cyc}} T^2(r_{ij}) T^2(r_{ik})$$

$$T(r) = \left(1 + \frac{3}{\mu r} + \frac{3}{\mu^2 r^2}\right) \frac{e^{-\mu r}}{\mu r} \left(1 - e^{-c_t r^2}\right)^2$$

$$\mu = \frac{(m_\pi^0 + 2m_\pi^{\mathrm{c}})}{3\hbar c} \qquad m_\pi^0 (m_\pi^{\mathrm{c}}) : \text{mass of } \pi^0 (\pi^c)$$

$U = 0.0048\mathrm{MeV}$, $\quad c_t = 2.1\mathrm{fm}^{-1}$,

$V_{ijk}^{2\pi}$: Two-pion exchange potential

$$V_{ijk}^{2\pi} = A \sum_{cyc} \left[\{\boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j, \boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_k\} \{x_{ij}, x_{ik}\} + \frac{1}{4} [\boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j, \boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_k][x_{ij}, x_{ik}] \right]$$

$$x_{ij} = T(r_{ij}) S_{\mathrm{T}ij} + Y(r_{ij})\, \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j$$

$$Y(r) = \frac{e^{-\mu r}}{\mu r} \left(1 - \mathrm{e}^{-c_t r^2}\right) \qquad A = -0.0293\mathrm{MeV}$$

[1] J. Carlson, V. R. Pandharipande and R. B. Wiringa, Nucl. Phys. A401 (1983) 59.

Quantum Many Body Theories

Brueckner theory

Perturbation theory: Scattering problem in nuclear matter

The bare NN interaction $\rightarrow$ G-matrix

Variational Method

The expectation value of the Hamiltonian is minimized
with respect to the form of the trial wave function.

Jastrow wave function Cluster expansion

Fermi Hypernetted Chain (FHNC): Partial resummation of cluster terms

Quantum Monte Carlo

The Schrödinger equation for large number of nucleons is solved directly.

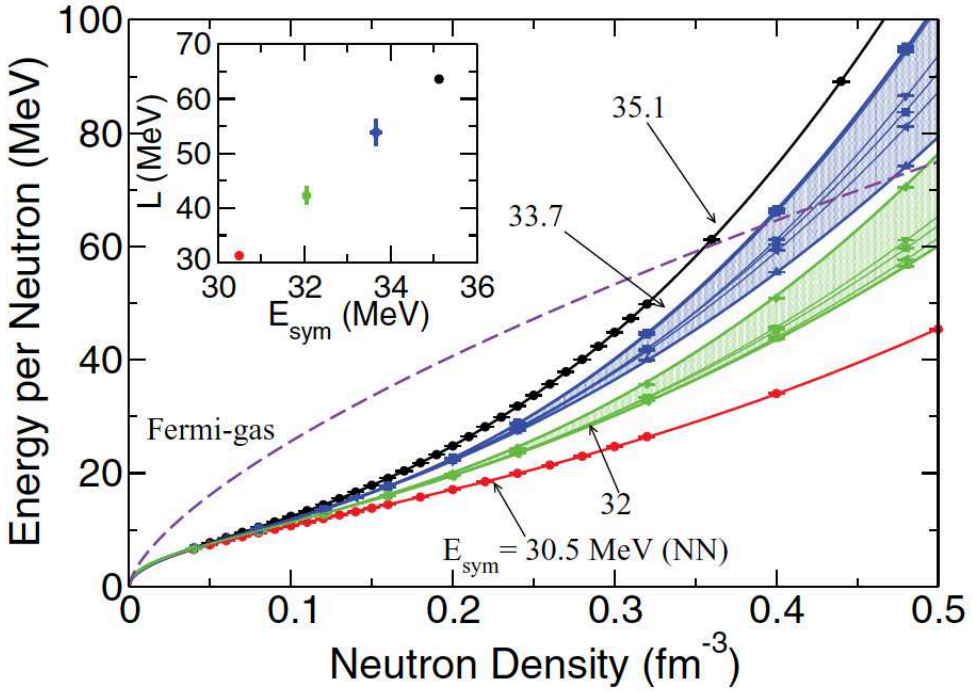
Auxiliary Field Diffusion Monte Carlo: AFDMC

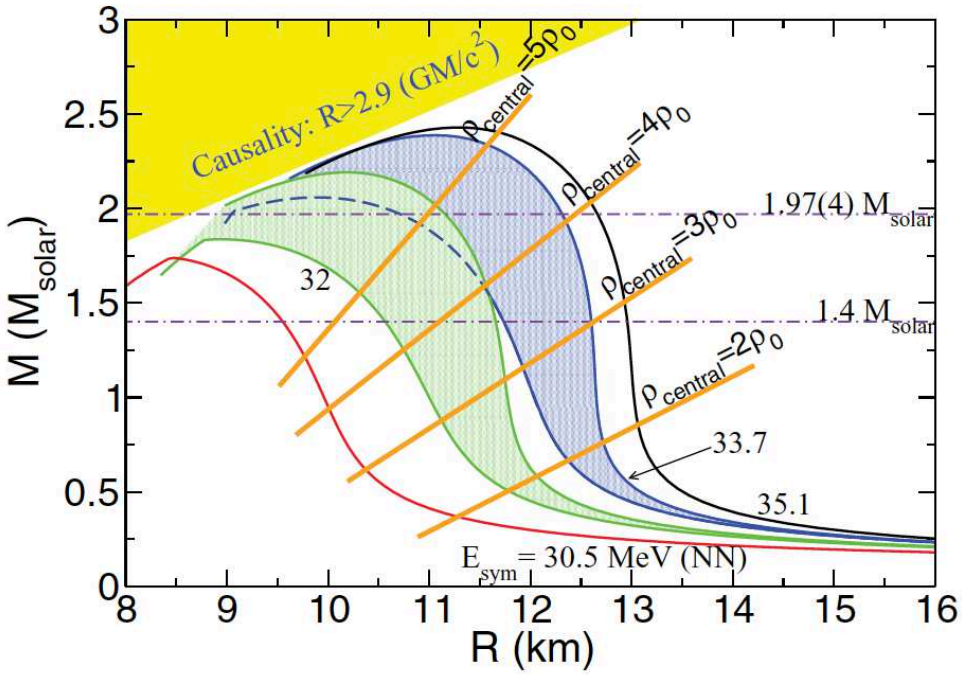
Green's Function Monte Carlo: GFMC

Maximum mass and radius of neutron stars, and the nuclear symmetry energy

S. Gandolfi,[1] J. Carlson,[1] and Sanjay Reddy[1,2]
[1]*Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA*
[2]*Institute for Nuclear Theory, University of Washington, Seattle, Washington 98195-1550, USA*
(Received 2 March 2011; published 9 March 2012)

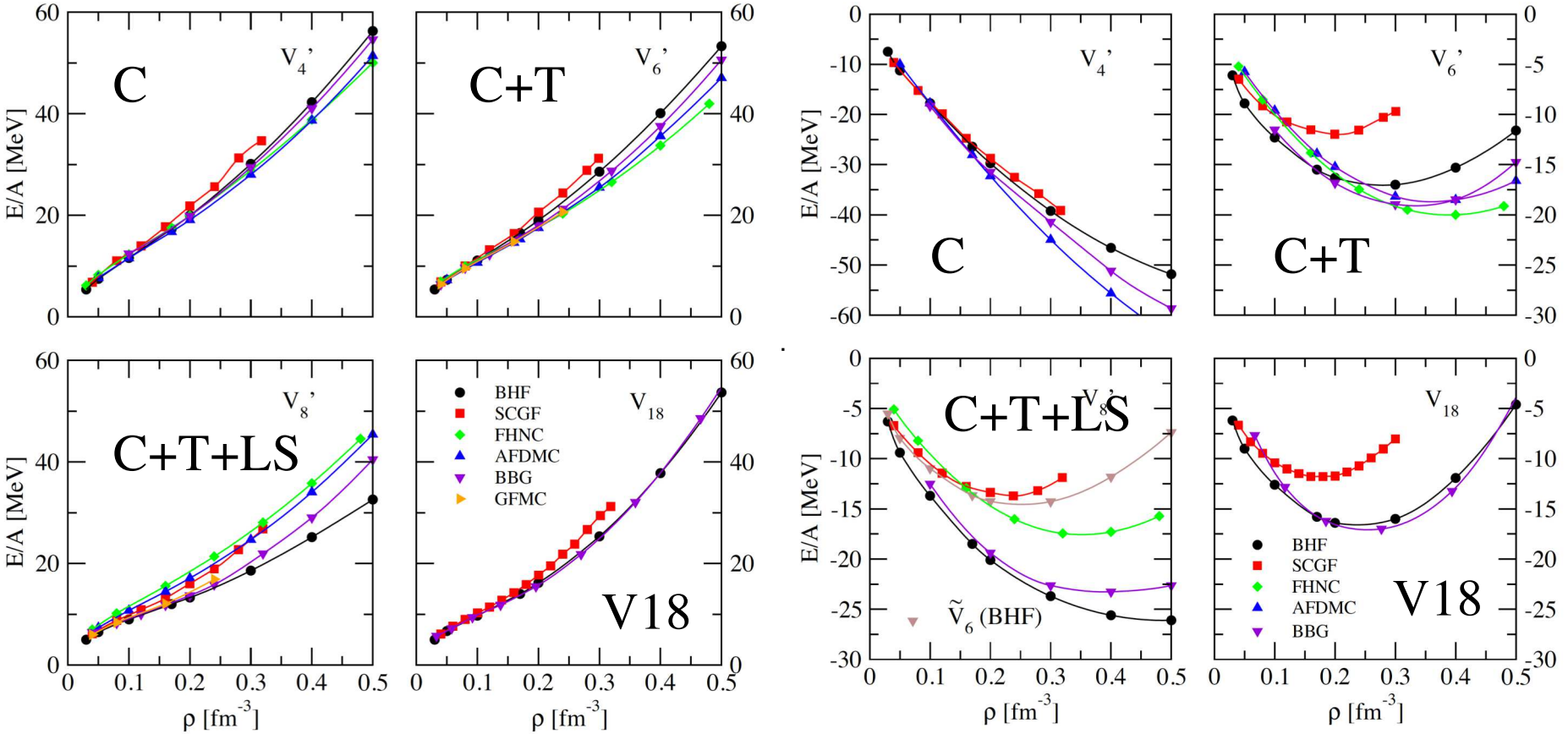
AV8'+IL (UIX) AFDMC calculations for neutron matter





Energies of Nuclear Matter (Two-Body Force)

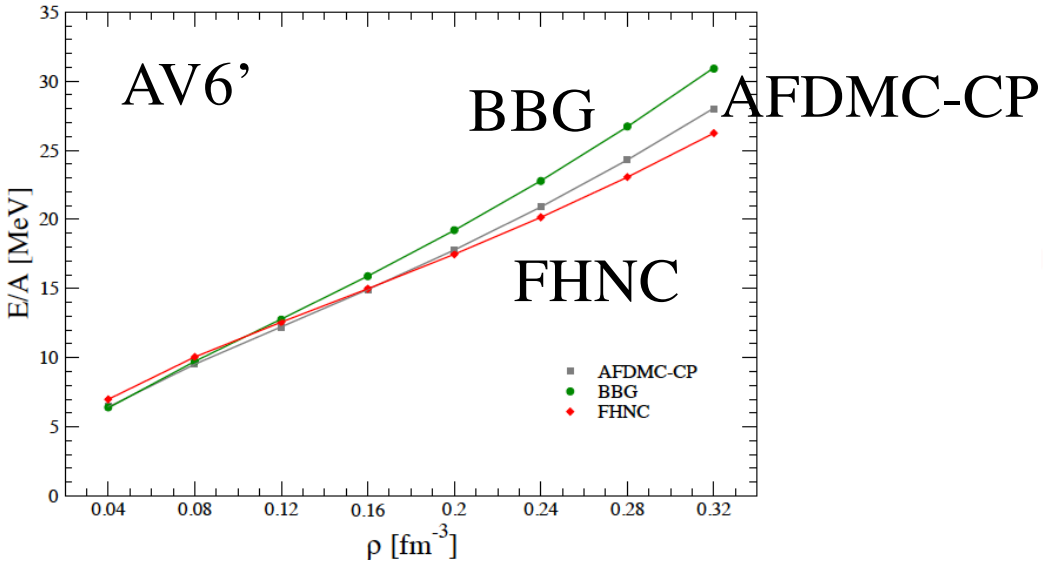
M. Baldo et al., PRC86 (2012) 064001.

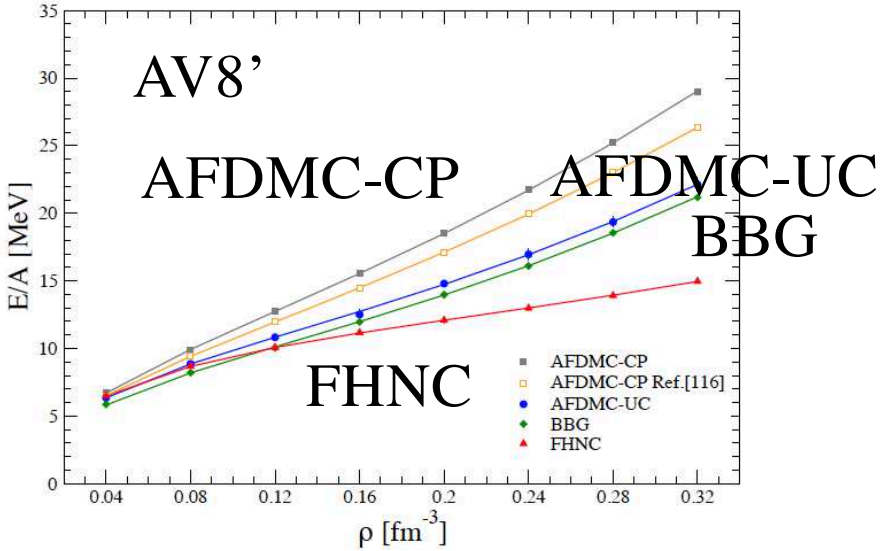


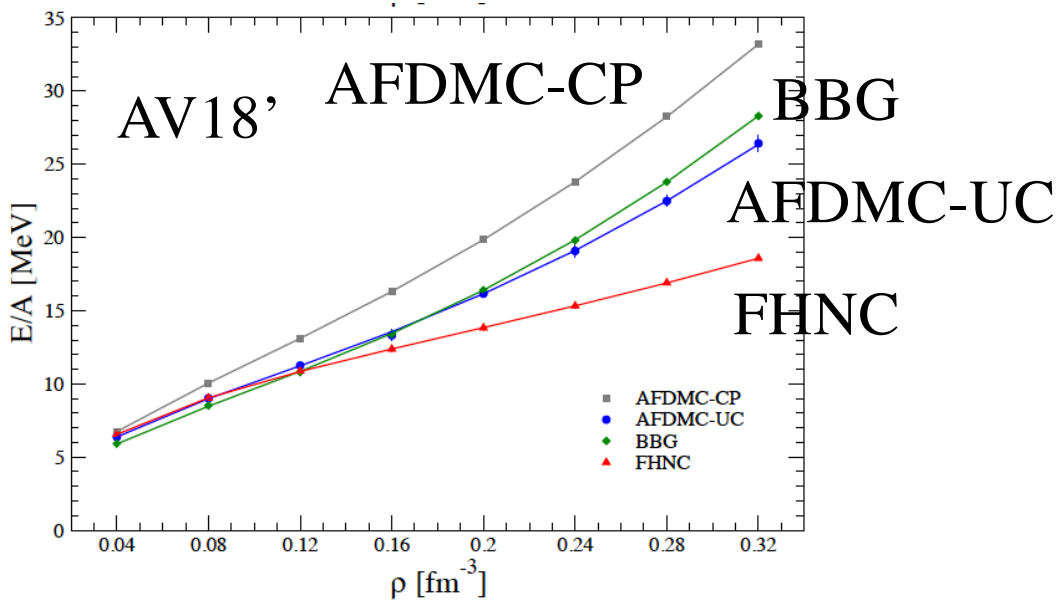
Pure Neutron Matter

Symmetric Nuclear Matter

Energy per neutron of pure neutron matter







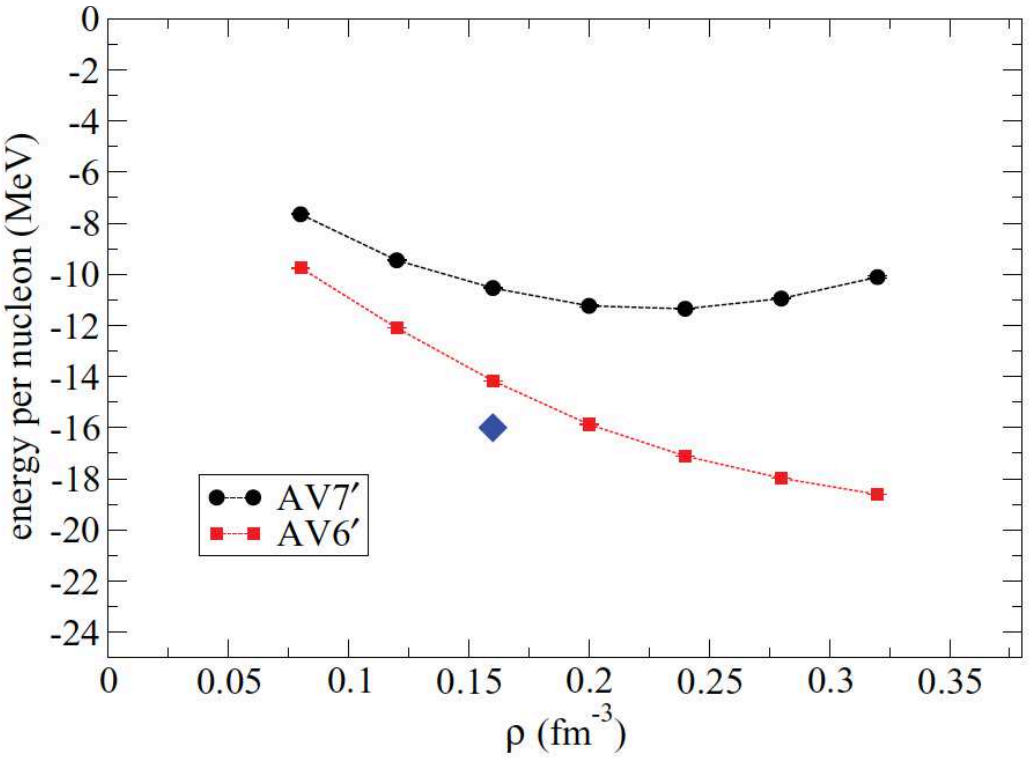
M. Piarulli, I. Bombaci, D. Logoteta, A. Lovato, and R. B. Wiringa, arXiv:1908.04426

AFDMC calculations for symmetric nuclear matter

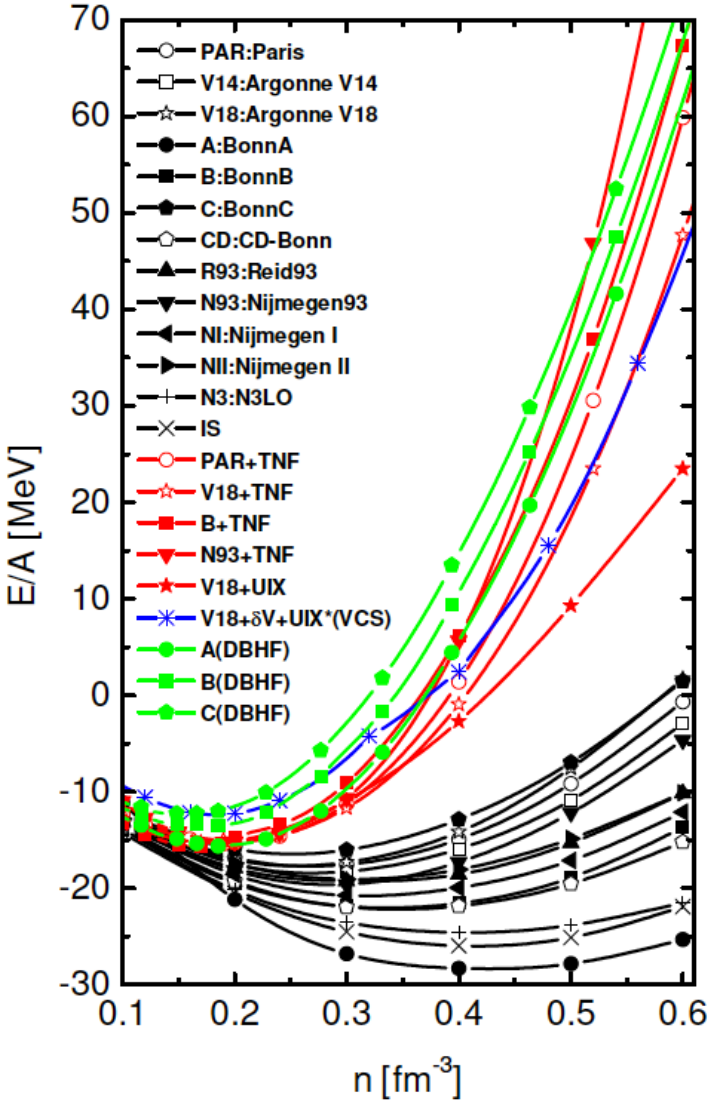
PHYSICAL REVIEW C **90**, 061306(R) (2014)

From the lightest nuclei to the equation of state of asymmetric nuclear matter with realistic nuclear interactions

S. Gandolfi,[1] A. Lovato,[2,3] J. Carlson,[1] and Kevin E. Schmidt[4]
[1]*Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA*
[2]*Argonne Leadership Computing Facility, Argonne National Laboratory, Argonne, Illinois 60439, USA*
[3]*Physics Division, Argonne National Laboratory, Argonne, Illinois 60439, USA*
[4]*Department of Physics, Arizona State University, Tempe, Arizona 85287, USA*
(Received 12 June 2014; published 29 December 2014)



Energies of symmetric nuclear matter



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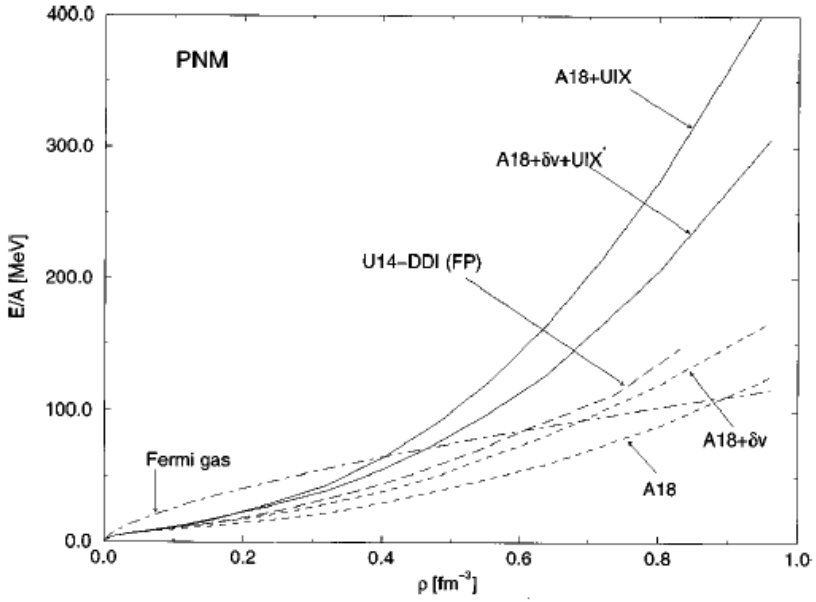
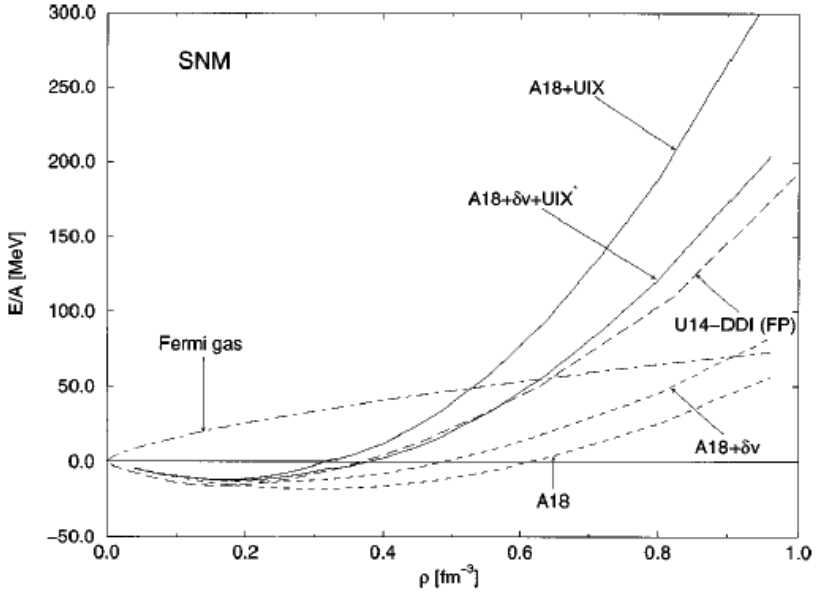
FHNC Calculations for Nuclear Matter (AV18+UIX)

A. Akmal, V. A. Pandharipande, and D. G. Ravenhall (APR)

Pure Neutron Matter

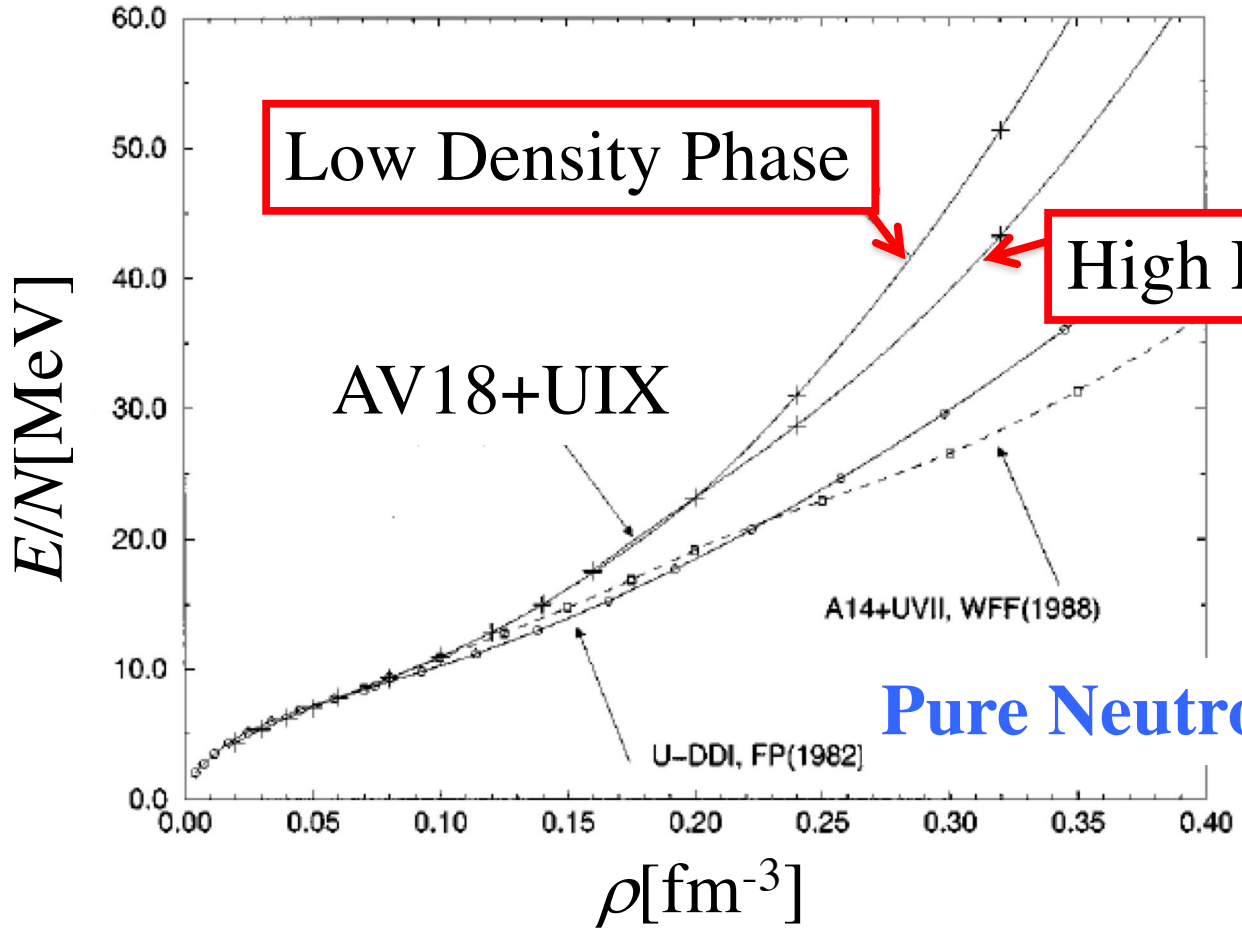


Symmetric Nuclear Matter



PRC **58** (1998)1804

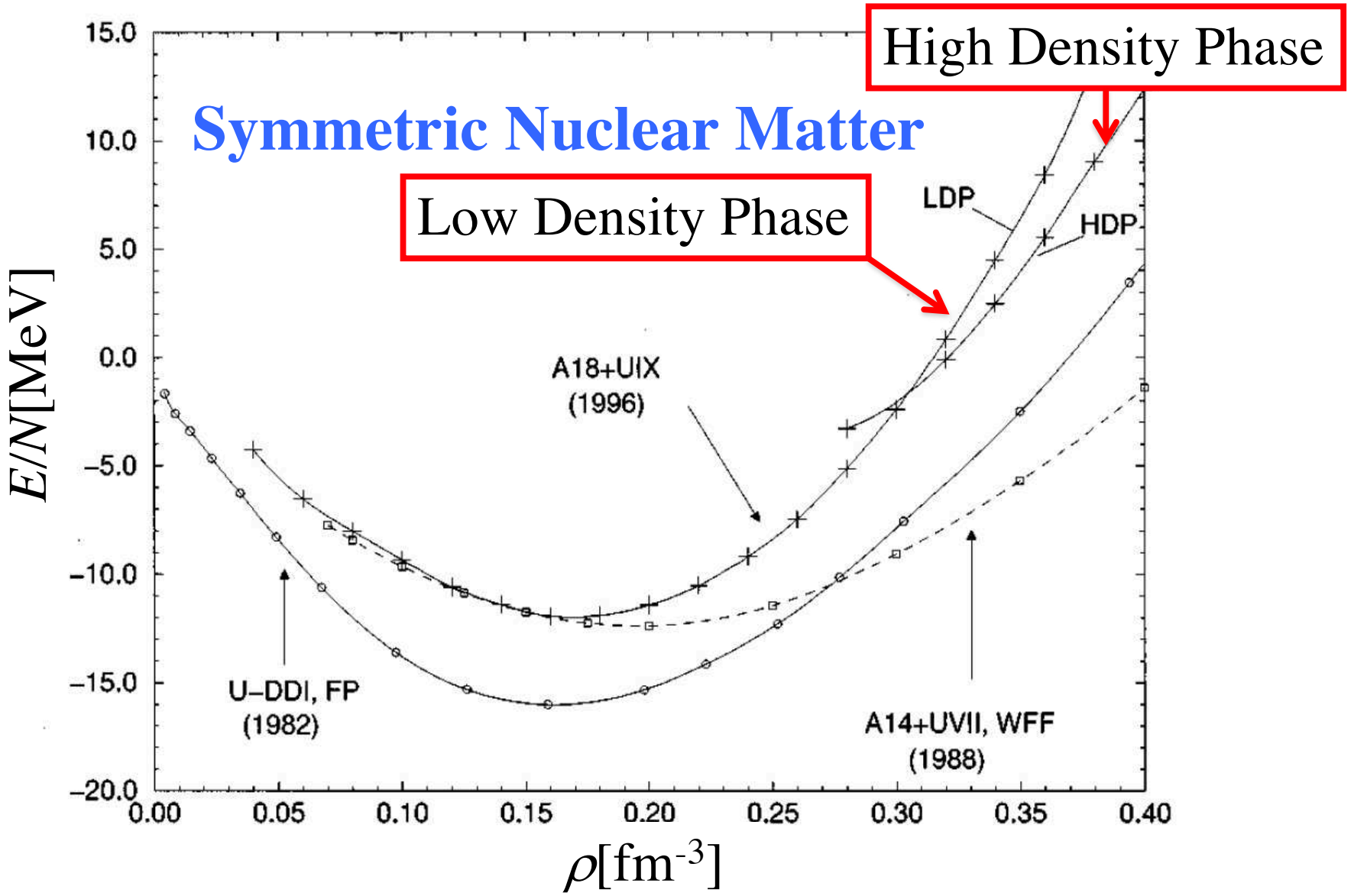
Pion condensation in the APR-EOS



2π-exchange three nucleon force induces π-condensation (1st order)

A. Akmal and V. A. Pandharipande, Phys. Rev. C56(1997)2262.

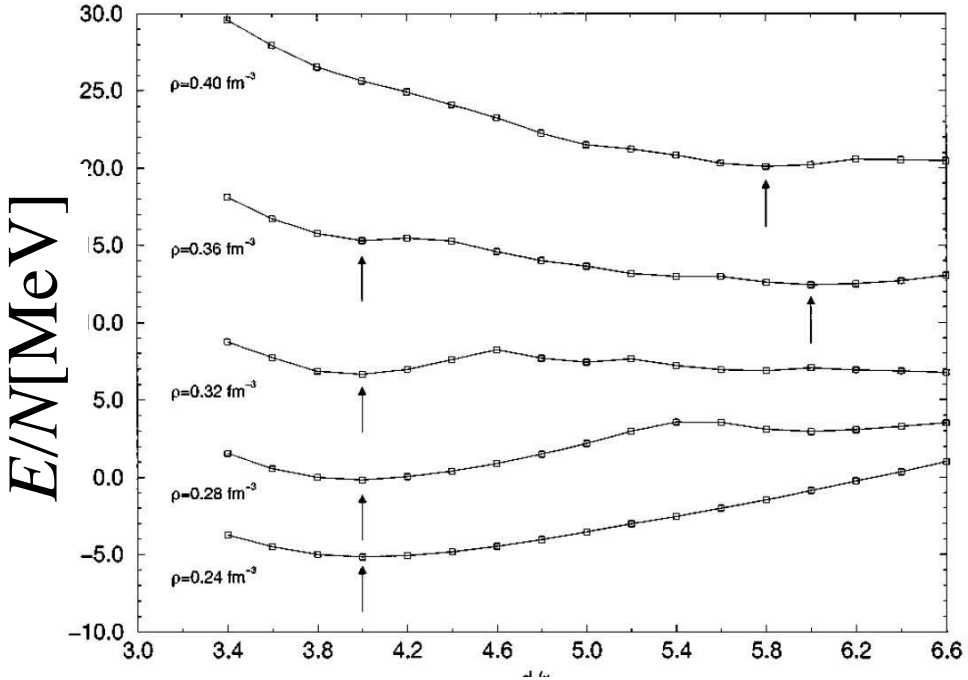
Pion condensation in the APR-EOS



A. Akmal and V. A. Pandharipande, Phys. Rev. C56(1997)2262.

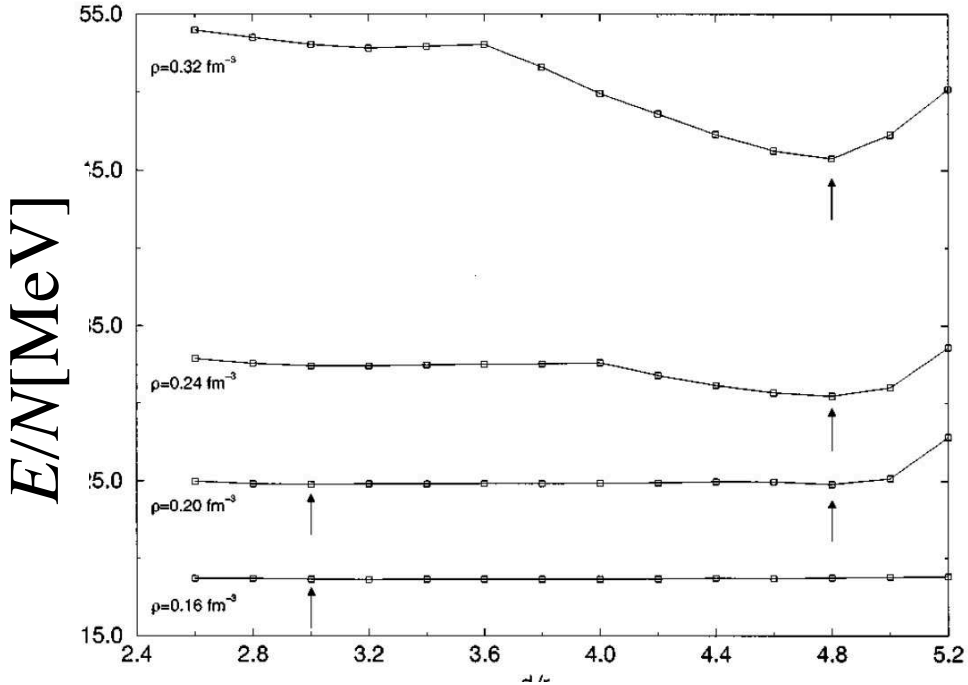
Jump of the tensor healing distances

Symmetric Nuclear Matter



Tensor healing distance d_t/r_0

Pure Neutron Matter



Tensor healing distance d_t/r_0

1st order phase transitions

A. Akmal and V. A. Pandharipande, Phys. Rev. C56(1997)2262.

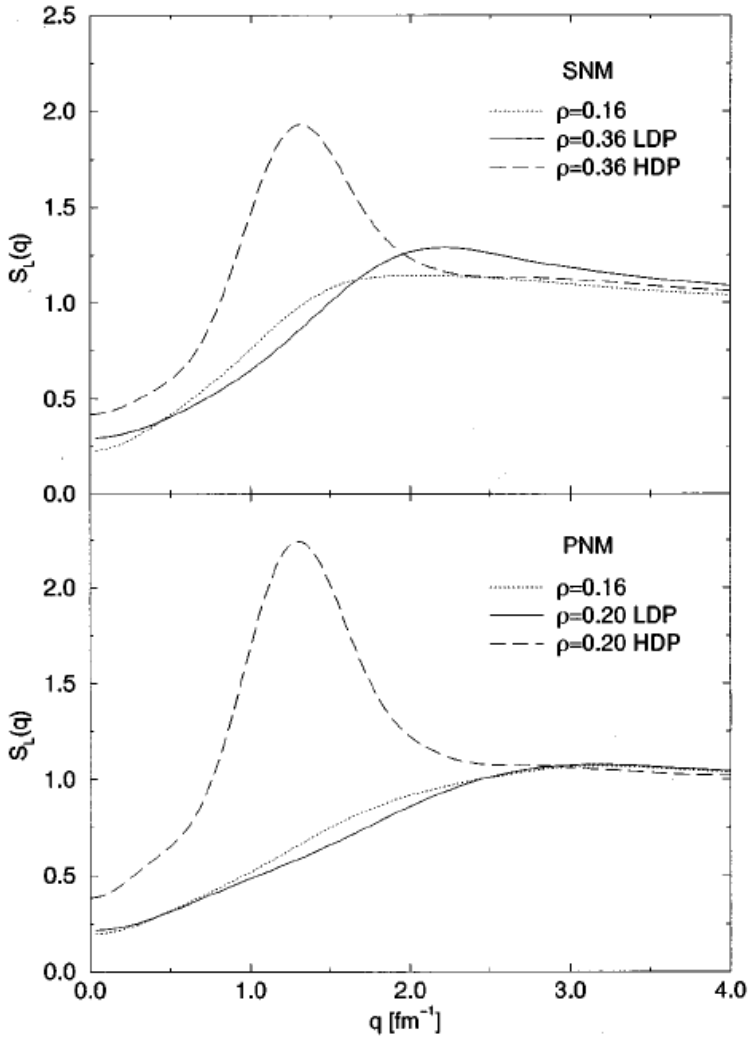


FIG. 11. $S_L(q)$ for SNM and PNM.

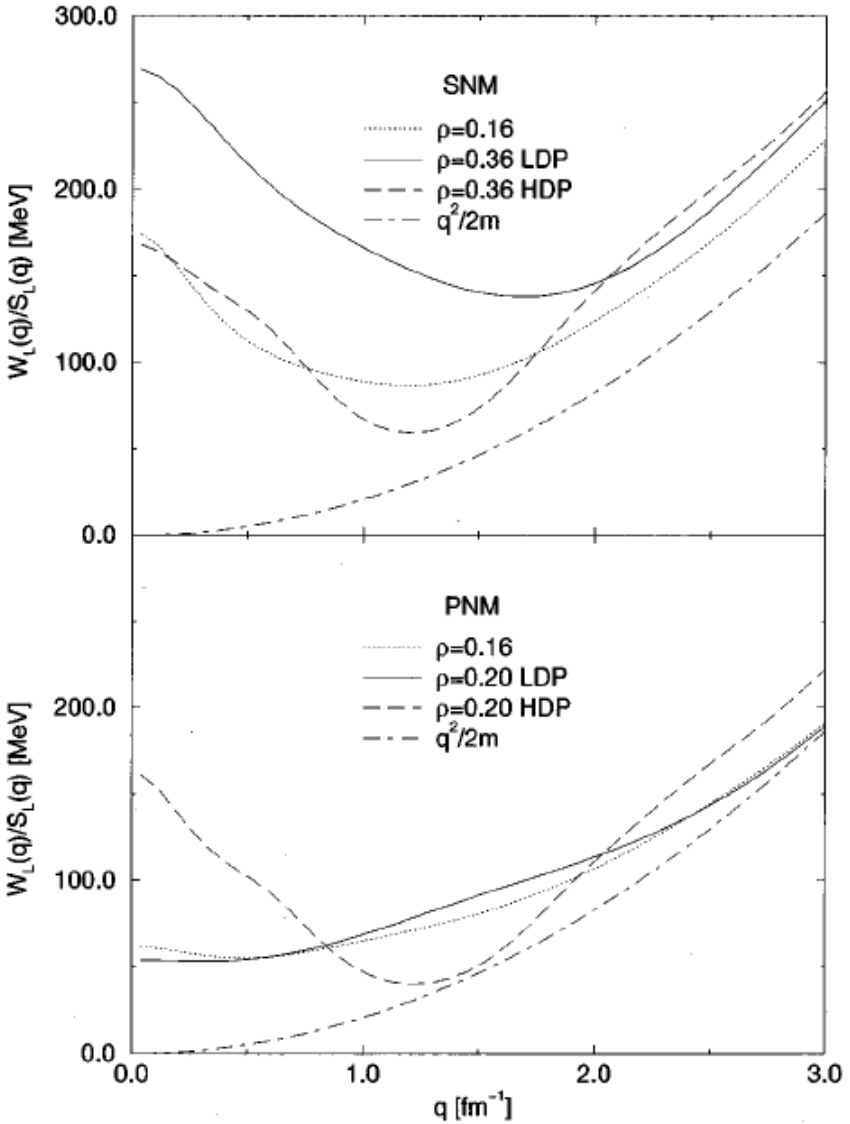


FIG. 12. $W_L(q)/S_L(q)$ for SNM and PNM. $q^2/2m$ (dash-dotted curve) is plotted for comparison.

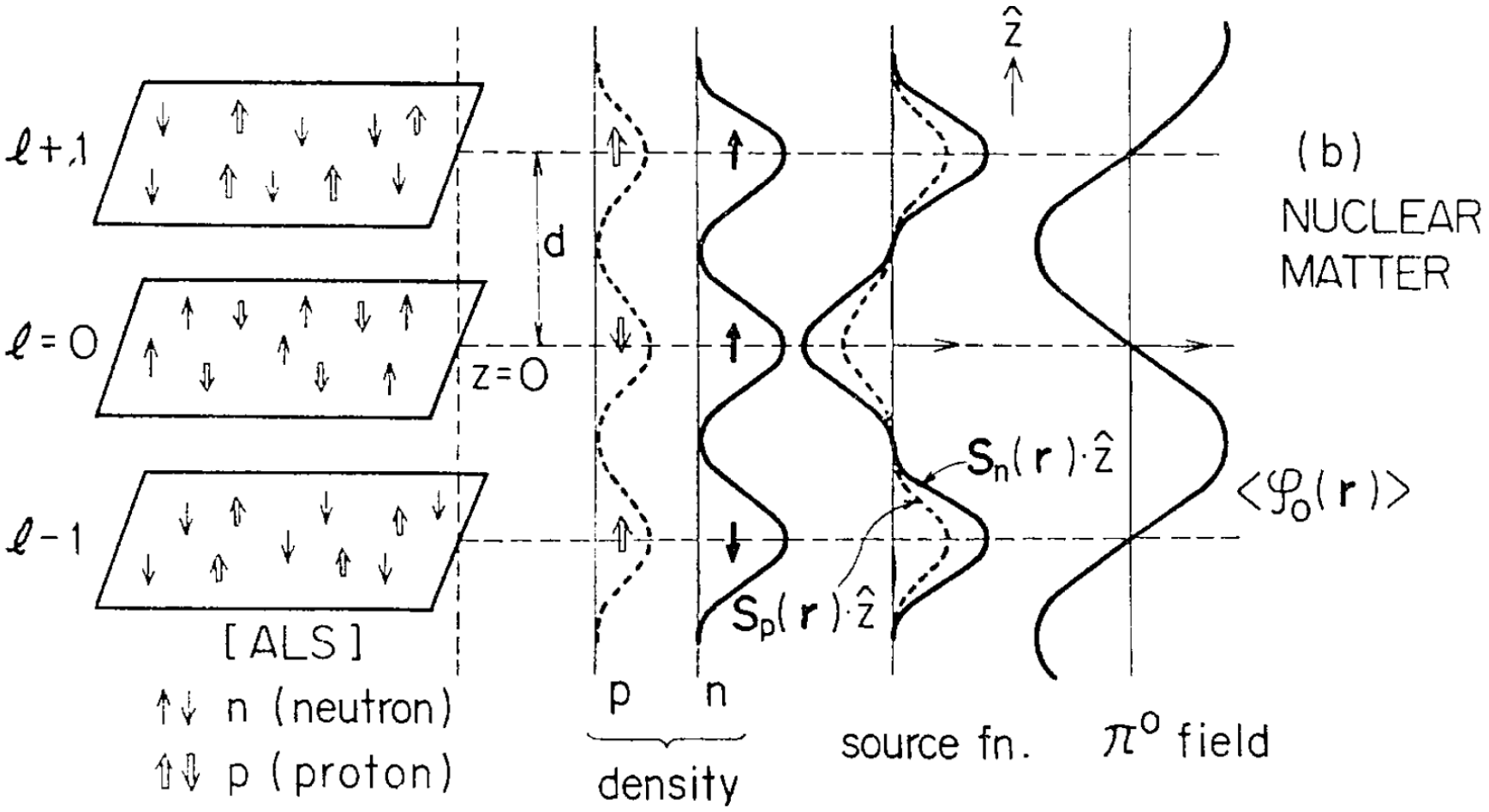
Structure functions by AP

Mean Energies of IVSL response by AP

Neutral Pion Condensation

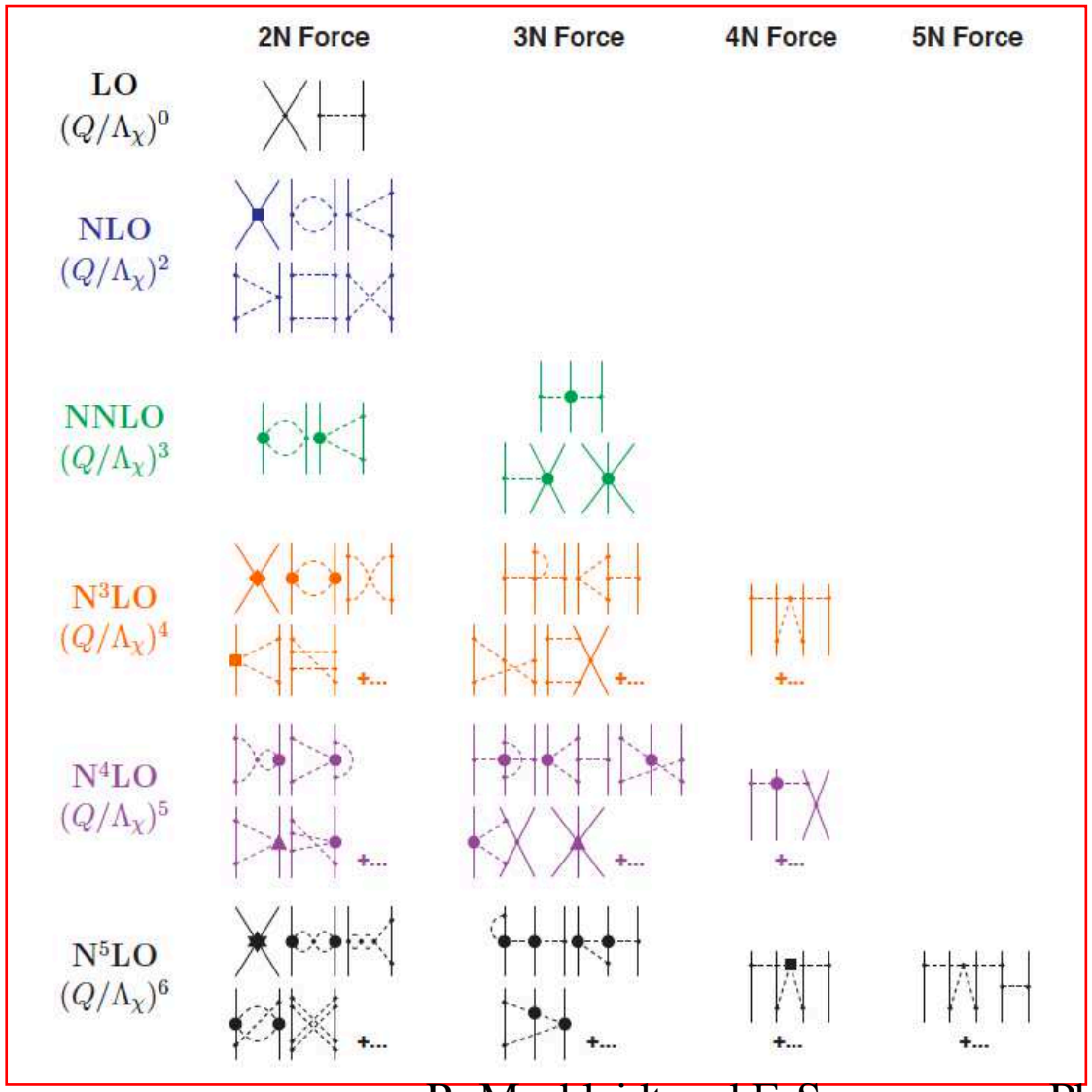
Alternating –Layer-Spin (ALS) Structure

T. Takatsuka et al., Prog. Theor. Phys. 59 (1978) 1933.



Nuclear Forces from the Chiral Effective Field Theory

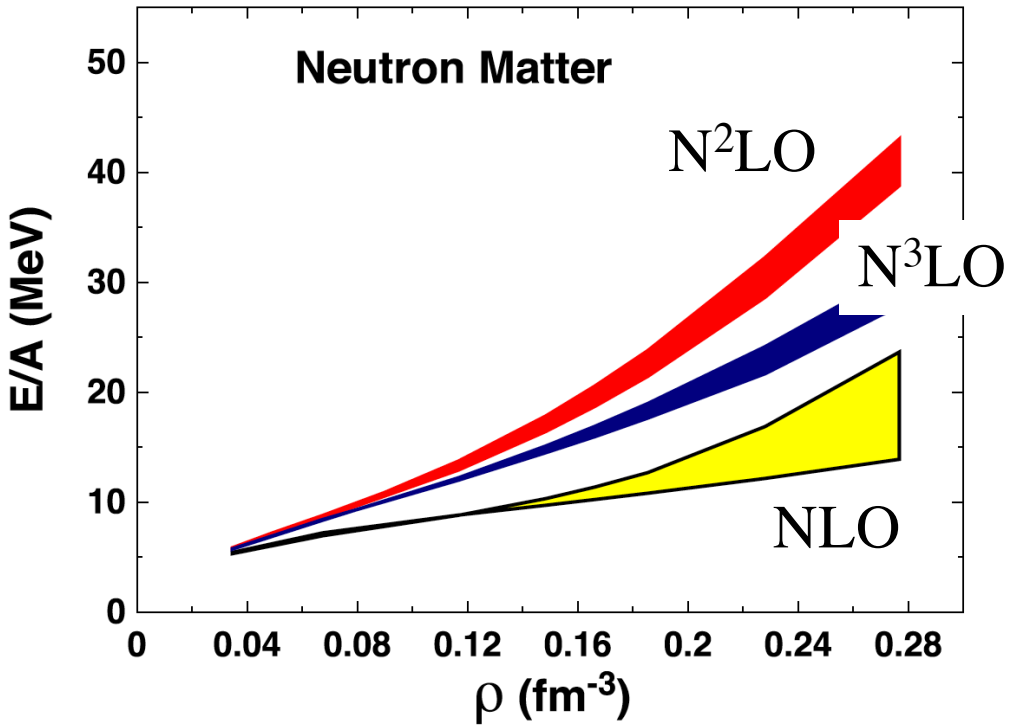
Based on the Chiral perturbation theory: EFT for low energy QCD



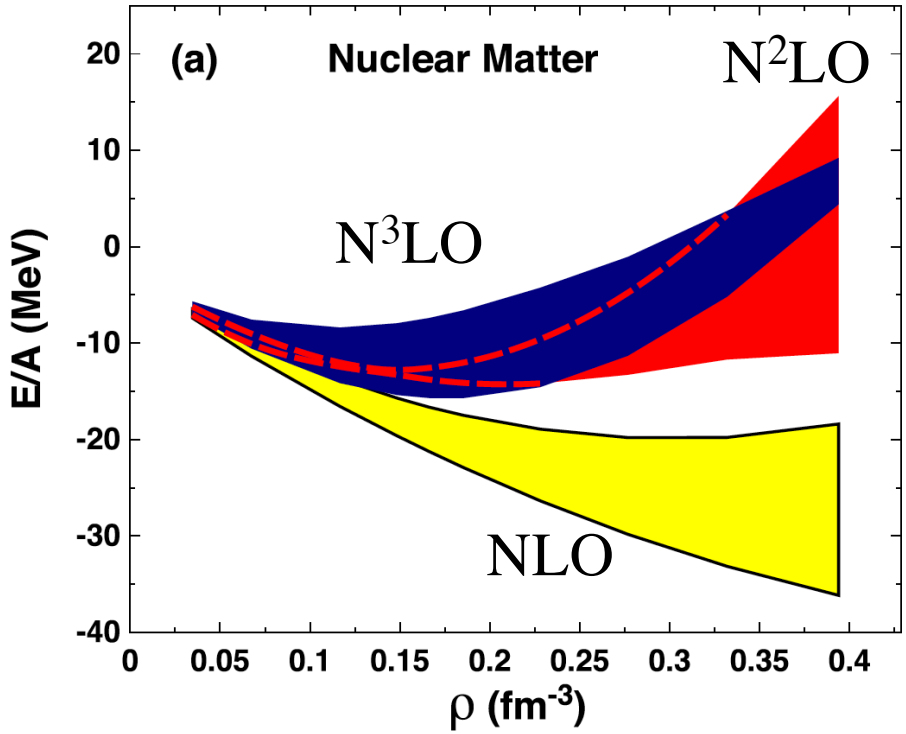
R. Machleidt and F. Sammarruca, Phys. Scr. 91 (2016) 083007.

Energies of nuclear matter with the χEFT

Pure Neutron Matter

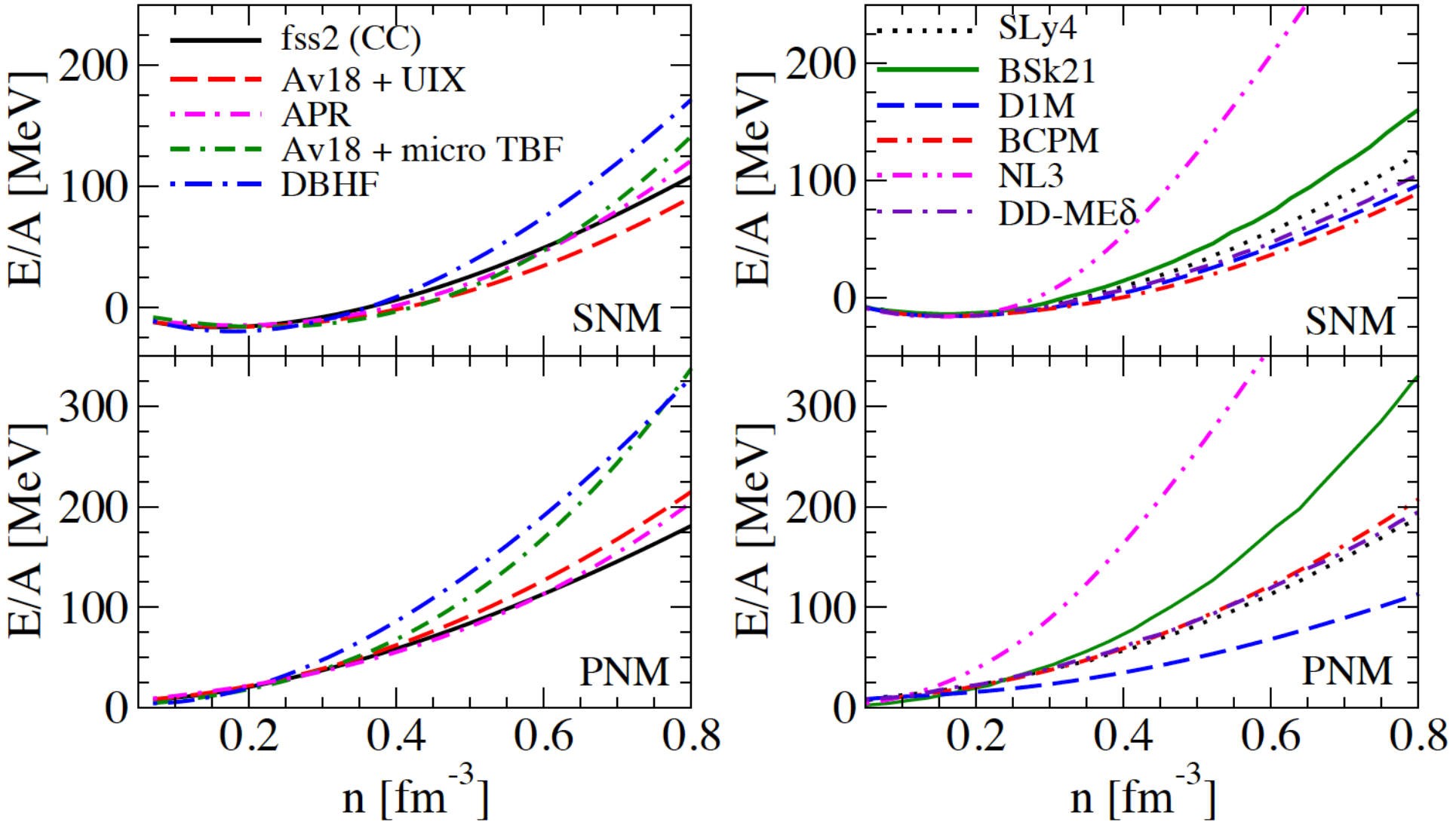


Symmetric Nuclear Matter



R. Machleidt and F. Sammarruca, Phys. Scr. 91 (2016) 083007.

Phenomenological nuclear EOSs



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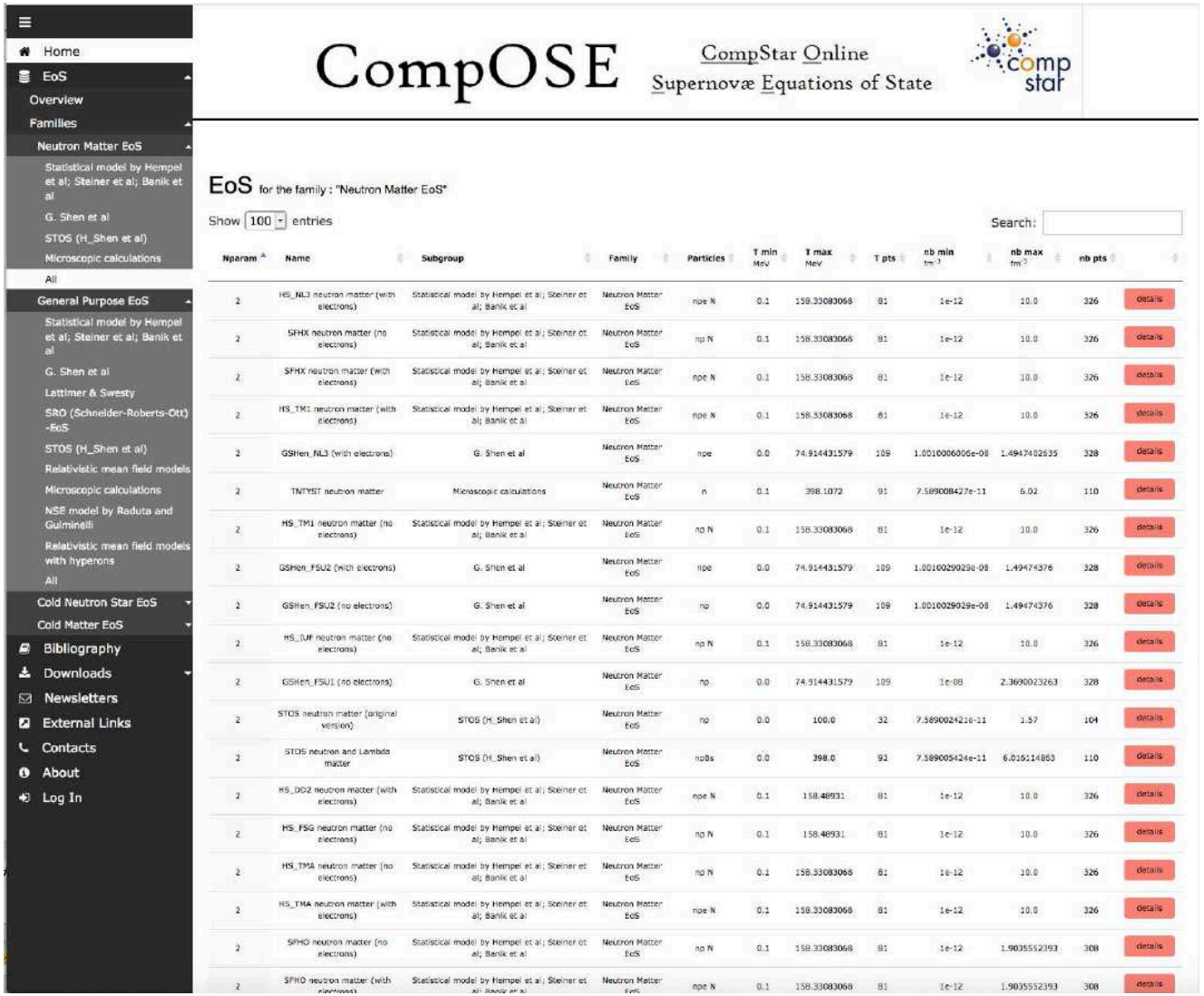
General Purpose EOS

CompOSE: https://compose.obspm.fr/



General Purpose EOS

Statistical model by Hempel et al.

G. Shen et al.

Lattimer and Swesty

Schneider-Roberts-Ott

STOS (H. Shen et al.)

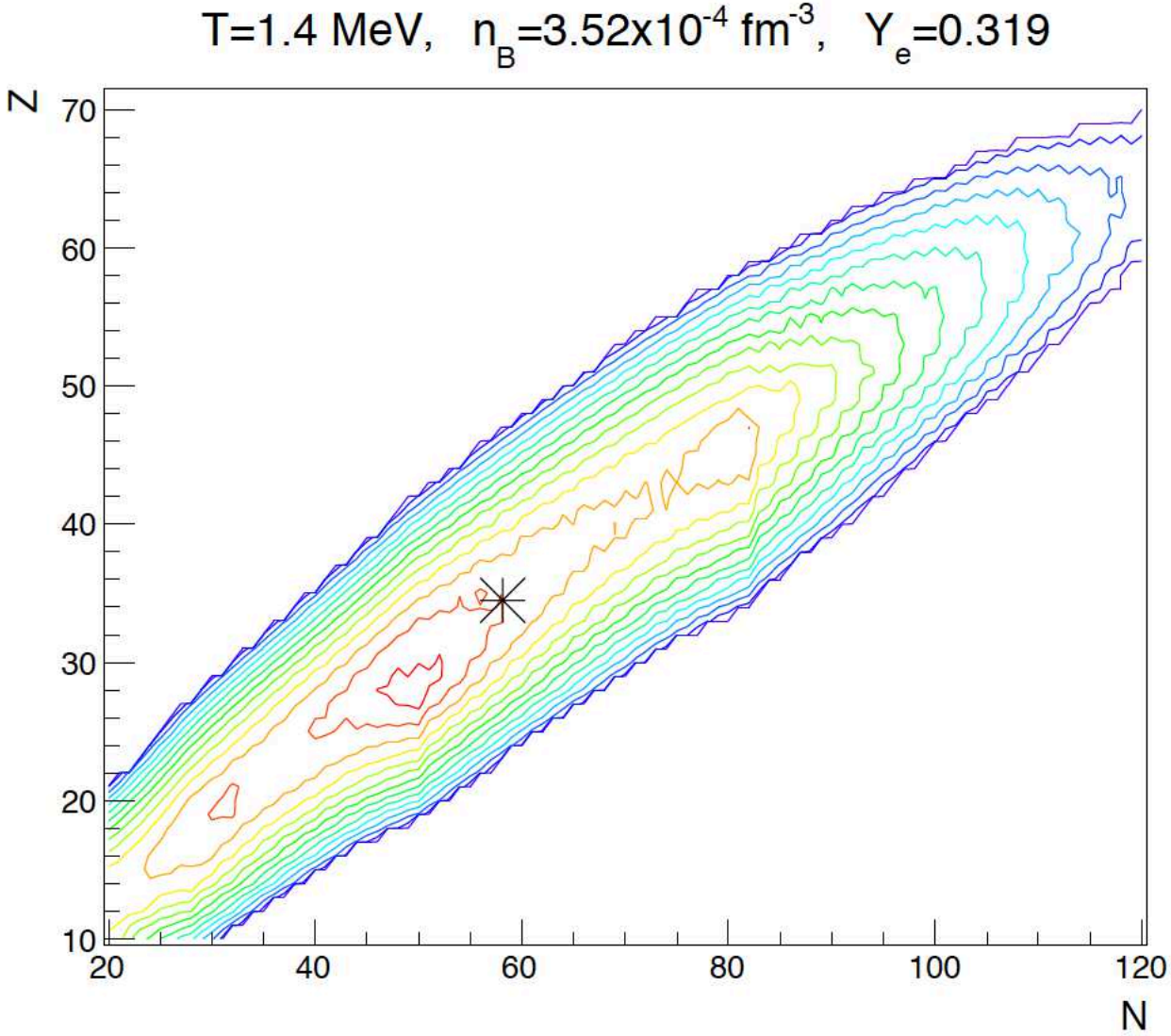
Relativistic mean field models

Microscopic calculations

NSE model by Raduta and Gulminelli

Relativistic mean field models with hyperons

Single nucleus approximation and nuclear statistical equilibrium



F. Culminelli and A. R. Raduta, Phys. Rev. C92 (2015) 055803

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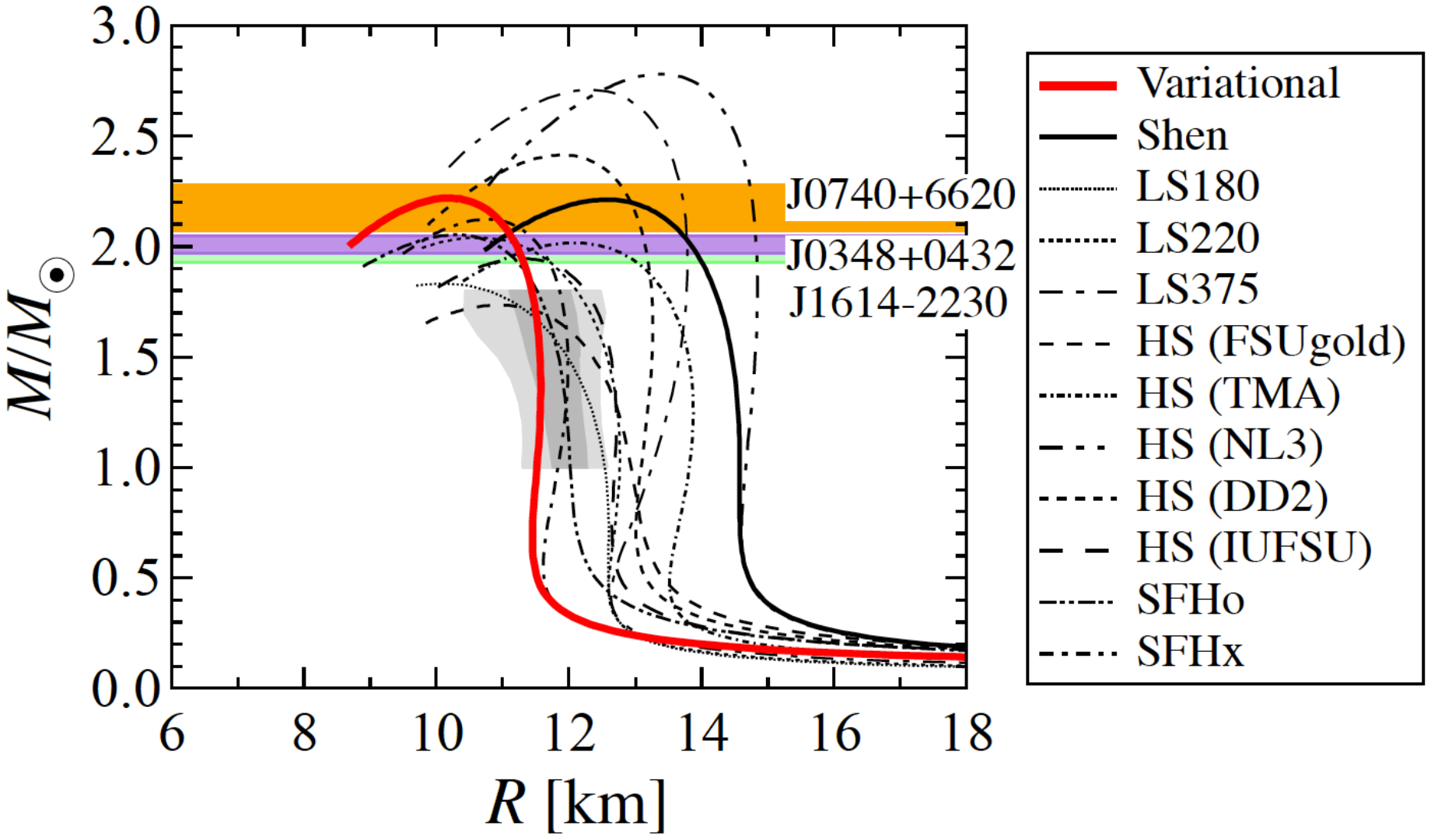
Masses and Radii of Neutron Stars



II. Cluster variational method for CCSNe

H. Togashi,[1] K. Nakazato,[2] Y. Takehara, [2]
S. Yamamuro, [2] H. Suzuki, [2] and M. Takano [3]

[1] Kyoshu Univ. [2] Tokyo Univ. of Sci., [3] Waseda Univ.

An EOS table for supernova numerical simulations
constructed with the cluster variational method
based on the Argonne v18 two-body potential
and the Urbana IX three-body potential

APR-EOS

Grid points

Parameter	Minimum	Maximum	Mesh	Number
$\log_{10}(T)$ [MeV]	-1.00	2.60	0.04	$91+1$
$Y_{\rm p}$	0.00	0.65	0.01	66
$\log_{10}(\rho_{\rm B})$ [g/cm^3]	5.1	16.0	0.10	110

*)H. Togashi et al., Nucl. Phys. A 961 (2017) 78.

Nuclear Hamiltonian

$$H = H_2 + H_3$$

$$H_2 = -\sum_{i=1}^{N} \frac{\hbar^2}{2m} \nabla_i^2 + \sum_{i<j}^{N} V_{ij} \quad , \qquad H_3 = \sum_{i<j<k}^{N} V_{ijk}$$

V_{ij}: The AV18 potential (isoscalar), V_{ijk}: The UIX potential

The AV18 two-body potential

R. B. Wiringa et al., PRC **51** (1995) 38

$$V_{ij} = \sum_{t=0}^{1} \sum_{s=0}^{1} \Big[V_{\mathrm{C}ts}(r_{ij}) + V_{\mathrm{T}t}(r_{ij}) S_{\mathrm{T}ij} + V_{\mathrm{SO}t}(r_{ij}) \left(\boldsymbol{s} \cdot \boldsymbol{L}_{ij} \right)$$

Central Tensor Spin-orbit

$$+ V_{\mathrm{qL}ts}(r_{ij}) \left| \boldsymbol{L}_{ij} \right|^2 + V_{\mathrm{qSO}t}(r_{ij}) \left(\boldsymbol{s} \cdot \boldsymbol{L}_{ij} \right)^2 \Big] P_{tsij} + V_{\mathrm{corr}}$$

Quadratic orbital angular momentum

Quadratic spin-orbit

t: isospin, s: spin

Experimental NN scattering data are well reproduced.

Jastrow wave function

$$\Psi\left(x_1,\cdots,x_N\right)=\mathrm{Sym}\left[\prod_{i<j}f_{ij}\right]\Phi_{\mathrm{F}}\left(x_1,\cdots,x_N\right)$$

Φ_{F}: The uncorrelated wave function Sym[ ]: Symmetrizer

f_{ij}: Two nucleon correlation function

$$f_{ij}=\sum_{t=0}^{1}\sum_{\mu}\sum_{s=0}^{1}\left[f_{\mathrm{C}ts}^{\mu}(r_{ij})+sf_{\mathrm{T}t}^{\mu}(r_{ij})S_{\mathrm{T}ij}+sf_{\mathrm{SO}t}^{\mu}(r_{ij})\left(\boldsymbol{s}\cdot\boldsymbol{L}_{ij}\right)\right]P_{tsij}^{\mu}$$

- $f_{\mathrm{C}ts}{}^{\mu}$: Central correlation function
- $f_{\mathrm{T}t}{}^{\mu}$: Tensor correlation function
- $f_{\mathrm{SO}t}{}^{\mu}$: Spin-orbit correlation function

$P_{tsij}{}^{\mu}$: Spin-isospin projection operator

t : isospin

s : spin

$\mu=(+,0,-)$

for (p-p, p-n, n-n) pairs (t =1)

Cluster variational method

The expectation value of the Hamiltonian

Cluster expansion

$$\frac{\langle H_2 \rangle}{N} = \frac{1}{N}\frac{\langle \Psi | H_2 | \Psi \rangle}{\langle \Psi | \Psi \rangle} = \frac{\langle H_2 \rangle_2}{N} + \frac{\langle H_2 \rangle_3}{N} + \cdots$$

Two-body cluster approximation

$\langle H_2 \rangle_2 / N$: Two-body cluster terms

$\langle H_2 \rangle_3 / N$: Three-body cluster terms

Taking into account
higher order cluster terms
FHNC method (APR, AM)

Two-body cluster approximation

$$\frac{E_2}{N} = \frac{\langle H_2 \rangle_2}{N}\left[f^{\mu}_{\mathrm{C}ts}(r), f^{\mu}_{\mathrm{T}t}(r), f^{\mu}_{\mathrm{SO}t}(r)\right]$$

$f_{\mathrm{C}ts}{}^{\mu}(r)$, $f_{\mathrm{T}t}{}^{\mu}(r)$, $f_{\mathrm{SO}t}{}^{\mu}(r)$ Euler-Lagrange equations

⟶ Minimized energy

$\frac{\langle H_2 \rangle}{N}$ Two-body cluster approximation

$$\frac{E_2}{N} = E_{\mathrm{F}}(x) + 2\pi\rho \sum_{t=0}^{1} \sum_{\mu} \sum_{s=0}^{1} \int \Big[F^{\mu}_{ts}(r) V_{\mathrm{C}ts}(r) + sF^{\mu}_{\mathrm{T}t}(r) V_{\mathrm{T}t}(r) + sF^{\mu}_{\mathrm{SO}t}(r) V_{\mathrm{SO}t}(r) + F^{\mu}_{\mathrm{qL}ts}(r) V_{\mathrm{qL}ts}(r) + sF^{\mu}_{\mathrm{qSO}t}(r) V_{\mathrm{qSO}t}(r) \Big] r^2 dr$$

$$+ \frac{2\pi\hbar^2\rho}{m} \sum_{t=0}^{1} \sum_{\mu} \sum_{s=0}^{1} \int \left[\left\{ \left[\frac{df^{\mu}_{\mathrm{C}ts}(r)}{dr} \right]^2 + 8s \left[\frac{df^{\mu}_{\mathrm{T}t}(r)}{dr} \right]^2 + 48s \left[\frac{f^{\mu}_{\mathrm{T}t}(r)}{r} \right]^2 \right\} F^{\mu}_{\mathrm{F}ts}(r) + \frac{2}{3} s \left[\frac{df^{\mu}_{\mathrm{SO}t}(r)}{dr} \right]^2 F^{\mu}_{\mathrm{qF}t}(r) \right] r^2 dr$$

$$F^{\mu}_{ts}(r) = \left[f^{\mu}_{\mathrm{C}ts}(r) \right]^2 F^{\mu}_{\mathrm{F}ts}(r) + 8s \left[f^{\mu}_{\mathrm{T}t}(r) \right]^2 F^{\mu}_{\mathrm{F}t1}(r) + \frac{2}{3} s \left[f^{\mu}_{\mathrm{SO}t}(r) \right]^2 F^{\mu}_{\mathrm{qF}t1}(r)$$

$$F^{\mu}_{\mathrm{T}t}(r) = 16 \left\{ f^{\mu}_{\mathrm{C}t1}(r) f^{\mu}_{\mathrm{T}t}(r) - \left[f^{\mu}_{\mathrm{T}t}(r) \right]^2 \right\} F^{\mu}_{\mathrm{F}t1}(r) - \frac{2}{3} s \left[f^{\mu}_{\mathrm{SO}t}(r) \right]^2 F^{\mu}_{\mathrm{qF}t1}(r)$$

$$F^{\mu}_{\mathrm{SO}t}(r) = -24 \left[f^{\mu}_{\mathrm{T}t}(r) \right]^2 F^{\mu}_{\mathrm{F}t1}(r) + \frac{4}{3} \left\{ f^{\mu}_{\mathrm{C}t1}(r) - \frac{1}{4} f^{\mu}_{\mathrm{SO}t}(r) - f^{\mu}_{\mathrm{T}t}(r) \right\} f^{\mu}_{\mathrm{SO}t}(r) F^{\mu}_{\mathrm{qF}t1}(r)$$

$$F^{\mu}_{\mathrm{qL}ts}(r) = \left[f^{\mu}_{\mathrm{C}ts}(r) \right]^2 F^{\mu}_{\mathrm{qF}ts}(r) + 8s \left[f^{\mu}_{\mathrm{T}t}(r) \right]^2 \left[6F^{\mu}_{\mathrm{F}t1}(r) + F^{\mu}_{\mathrm{qF}ts}(r) \right] + \frac{2}{3} s \left[f^{\mu}_{\mathrm{SO}t}(r) \right]^2 F^{\mu}_{\mathrm{bF}t1}(r)$$

$$F^{\mu}_{\mathrm{qSO}t}(r) = \frac{2}{3} \left[f^{\mu}_{\mathrm{C}t1}(r) \right]^2 F^{\mu}_{\mathrm{qF}t1}(r) - \frac{2}{3} f^{\mu}_{\mathrm{C}t1}(r) \left[2f^{\mu}_{\mathrm{T}t}(r) + f^{\mu}_{\mathrm{SO}t}(r) \right] F^{\mu}_{\mathrm{qF}t1}(r)$$

$$+ 8s \left[f^{\mu}_{\mathrm{T}t}(r) \right]^2 \left[72F^{\mu}_{\mathrm{F}t1}(r) + \frac{20}{3} F^{\mu}_{\mathrm{qF}t1}(r) \right] + \frac{8}{3} f^{\mu}_{\mathrm{T}t}(r) f^{\mu}_{\mathrm{SO}t}(r) F^{\mu}_{\mathrm{qF}t1}(r) + \frac{2}{3} s \left[f^{\mu}_{\mathrm{SO}t}(r) \right]^2 F^{\mu}_{\mathrm{bF}t1}(r)$$

Two constraints

1. Mayer condition (Normalization conditions)

$$4\pi\rho\int_0^\infty \left[F_{ts}^{\mu}(r) - F_{\mathrm{F}ts}^{\mu}(r)\right] r^2 dr = 0$$

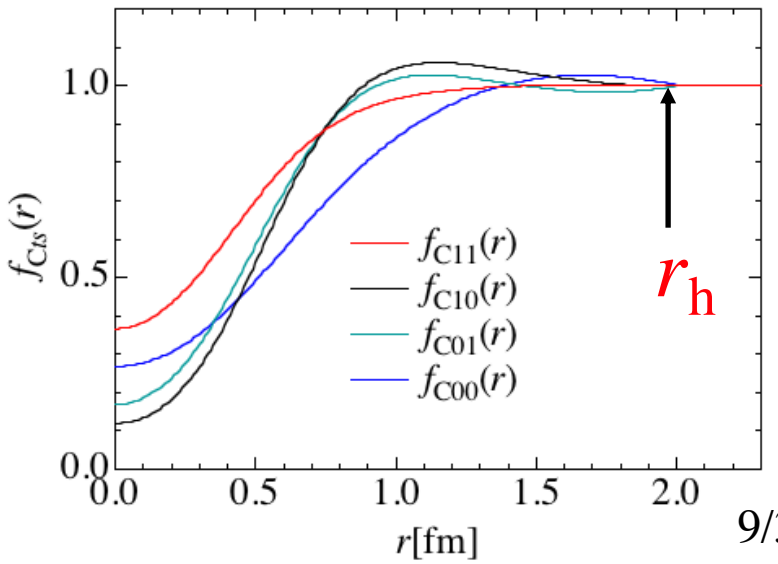
2. Healing distance: r_{h}

$$f_{\mathrm{C}ts}{}^{\mu}(r) = 1,\ f_{\mathrm{T}t}{}^{\mu}(r) = 0,\ f_{\mathrm{SO}t}{}^{\mu}(r) = 0 \quad (r > r_{\mathrm{h}})$$

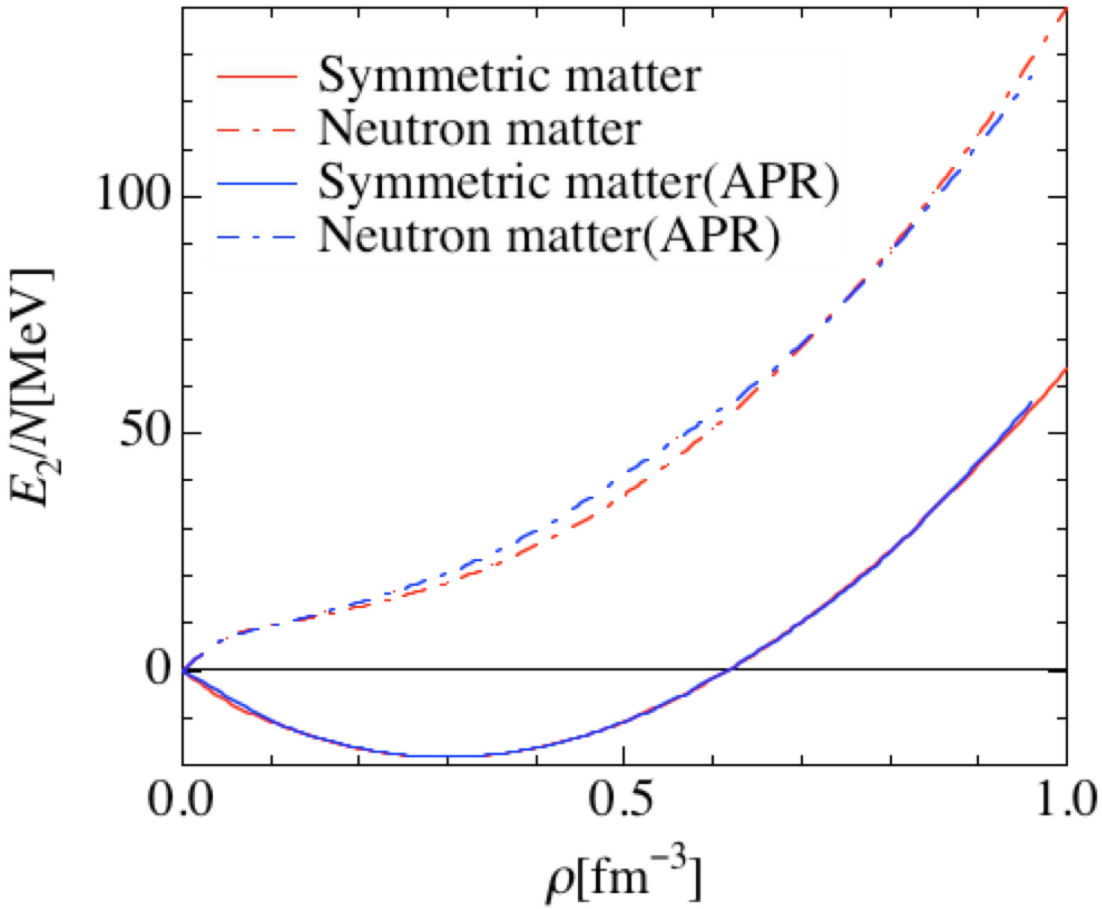
$$\boxed{r_{\mathrm{h}} = a_{\mathrm{h}}\, r_0} \qquad \frac{4\pi r_0^3}{3} = \frac{1}{\rho}$$

a_{h}: Adjustable parameter

The value of a_{h} is chosen so that the energy of symmetric nuclear matter reproduces the result by APR (FHNC).



E_2/N：Two-body energy



Kanzawa et al.
NPA791(2007)232

$r_h = a_h r_0$

$a_h = 1.76$

Two-body approximation + Healing distance condition

⇩

The result by APR(FHNC calculation) is reproduced.

*) A. Akmal et al., Phys. Rev. C58(1998)1804

Energies of three-body forces

UIX potential $V_{ijk} = V_{ijk}^{2\pi} + V_{ijk}^{\mathrm{R}}$

$V_{ijk}^{2\pi}$: *2π exchange term* V_{ijk}^{R} : Repulsive term

Expectation values with the degenerate Fermi gas wave function

$$\frac{E_3^{2\pi}}{N} = \frac{1}{N}\sum_{i<j<k}^{N} \left\langle V_{ijk}^{2\pi} \right\rangle_{\mathrm{F}} \qquad \frac{E_3^{\mathrm{R}}}{N} = \frac{1}{N}\sum_{i<j<k}^{N} \left\langle V_{ijk}^{\mathrm{R}} \right\rangle_{\mathrm{F}}$$

$$\frac{E_3}{N}(x) = \alpha \frac{E_3^{\mathrm{R}}}{N}(x) + \beta \frac{E_3^{2\pi}}{N}(x) + \gamma\rho^2 \mathrm{e}^{-\rho\delta}\left[1 - (1-2x)^2\right]$$

Correction term

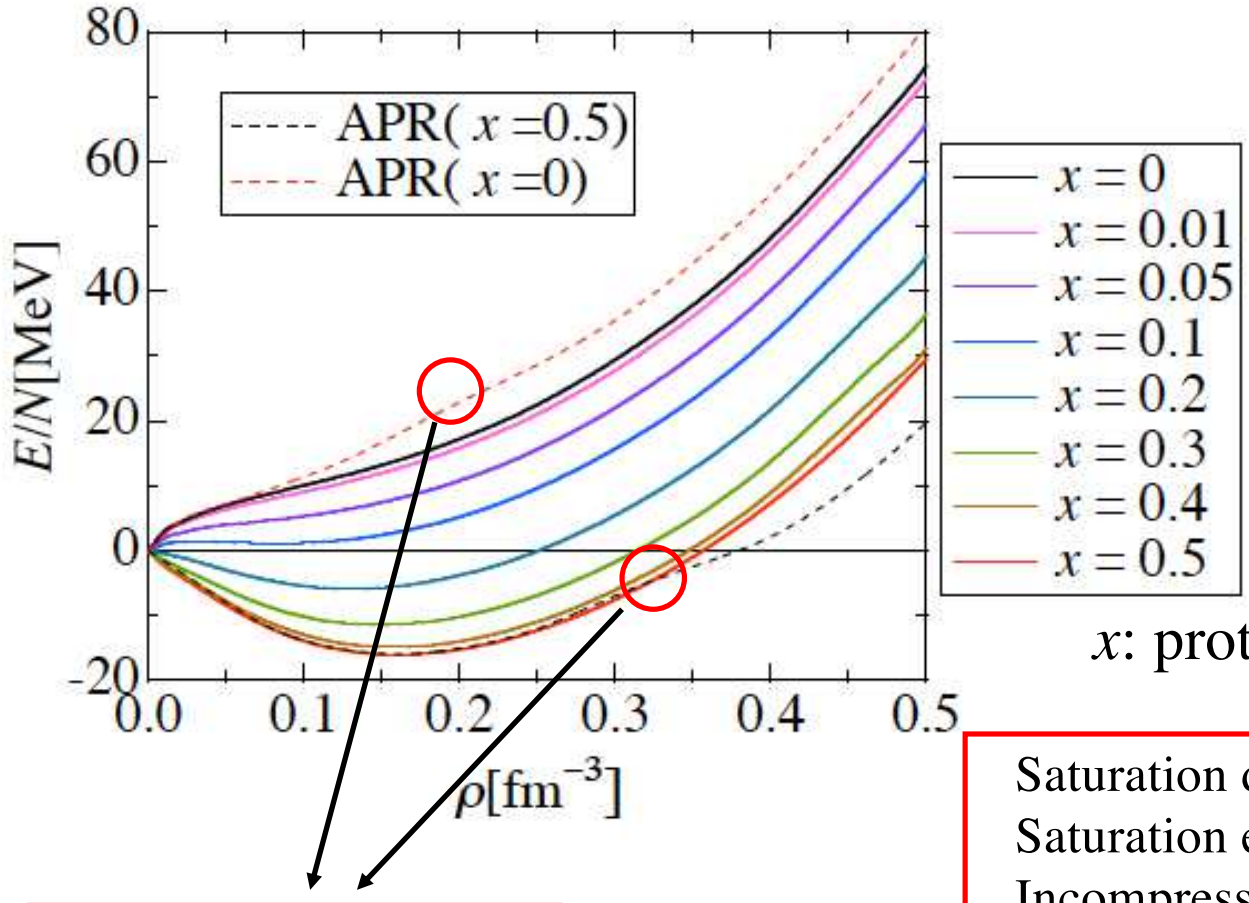
Total energy $\dfrac{E}{N} = \dfrac{E_2}{N} + \dfrac{E_3}{N}$

x: Ptoron fraction

Adjustable parameters : $\alpha, \beta, \gamma, \delta$

Chosen so as to reproduce the empirical saturation point

Energy of asymmetric nuclear matter at zero temperature

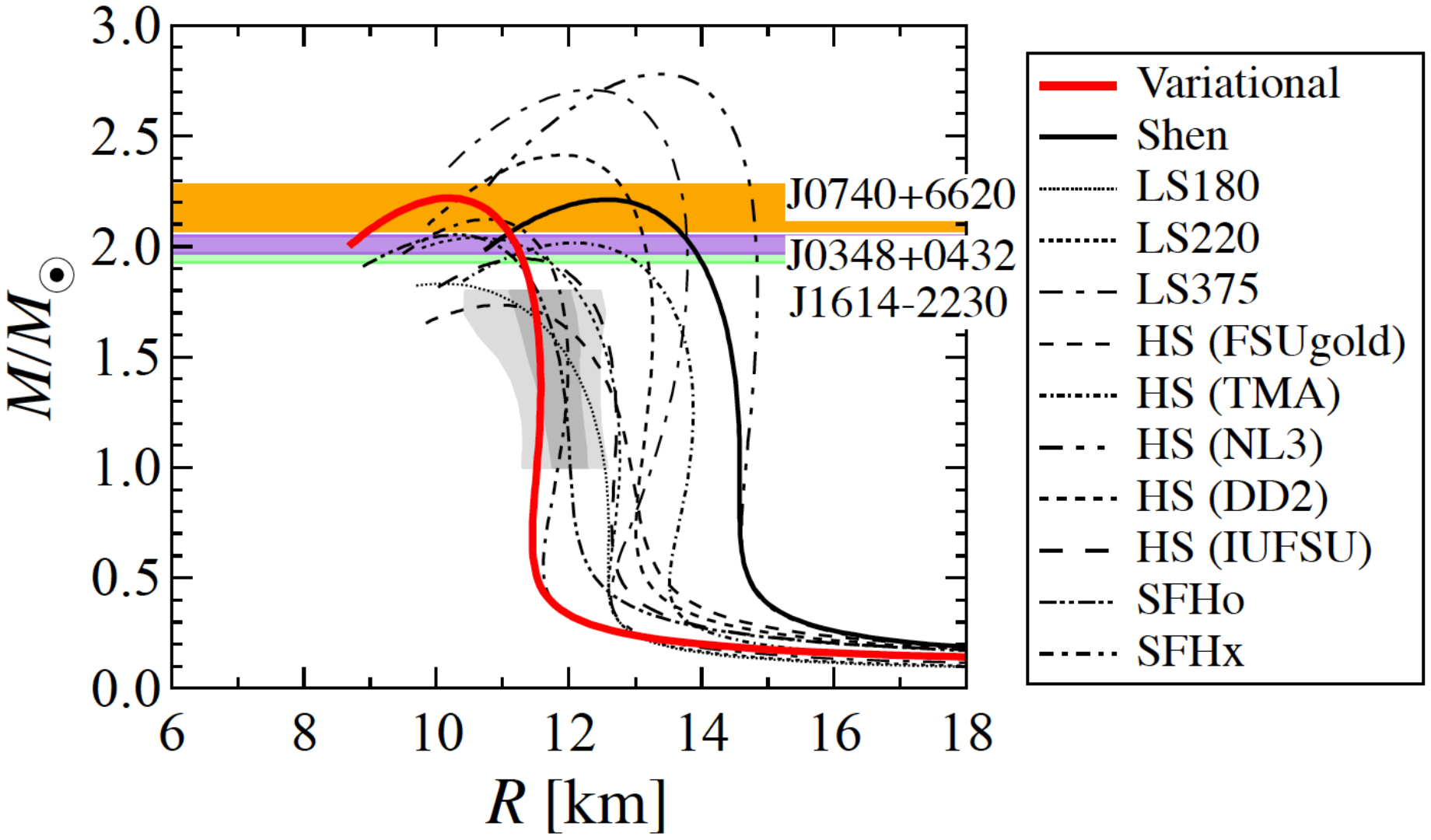


x: proton fraction

Pion condensation

Saturation density: 0.16 fm^{-3}
Saturation energy: –16.1 MeV
Incompressibility: 250 MeV
Symmetry energy: 30 MeV

Masses and Radii of Neutron Stars



The EOS of nuclear matter at finite temperature

K. E. Schmidt and V. R. Pandharipande: Phys. Lett. 87B(1979) 11.

Free energy $\frac{F}{N} = \frac{E_0}{N} - T\frac{S_0}{N}$

E_0/N : Internal energy

S_0/N : Entropy

$$\frac{E_0}{N} = \frac{E_2}{N} + \frac{E_3}{N}$$

E_2/N: In the E/N at zero temperature, single-nucleon occupation probabilities are replaced by $n_i(k)$ at temperature T.

E_3/N: The same as at zero temperature.

Averaged occupation probabilities $n_i(k)$

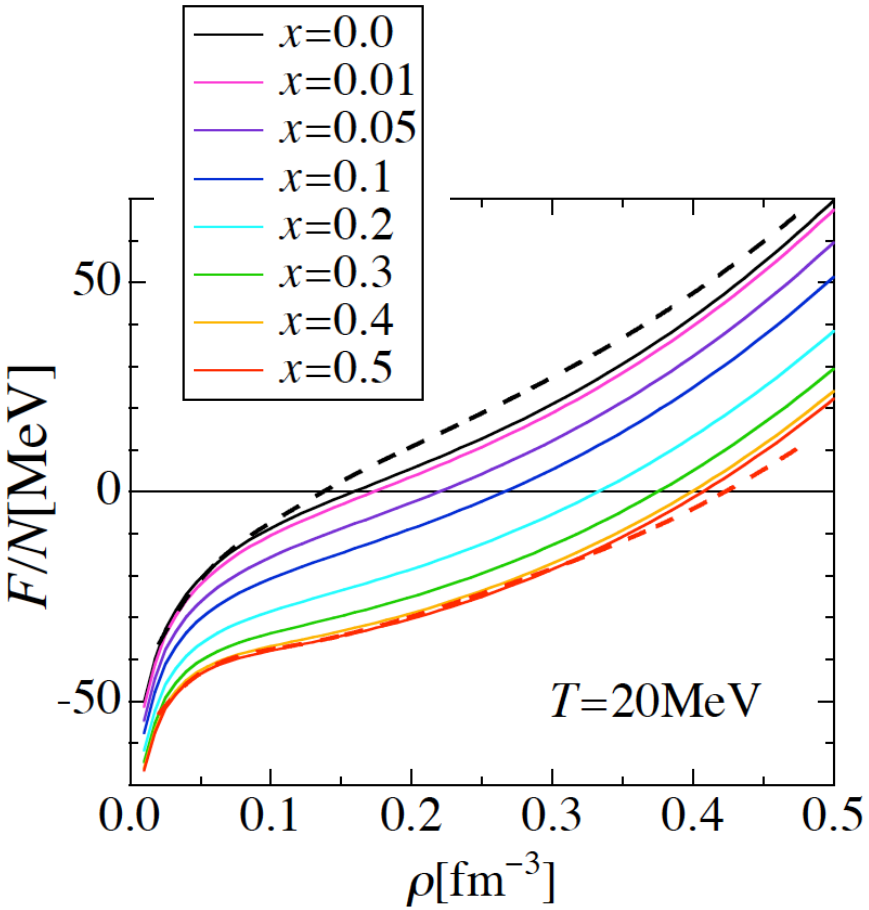
$$n_i(k) = \left\{1 + \exp\left[\left[e_i(k) - \mu_i\right]/k_B T\right]\right\}^{-1}, \quad e_i(k) = \frac{\hbar^2 k^2}{2m_i^*} \quad (i = \mathrm{p}, \mathrm{n})$$

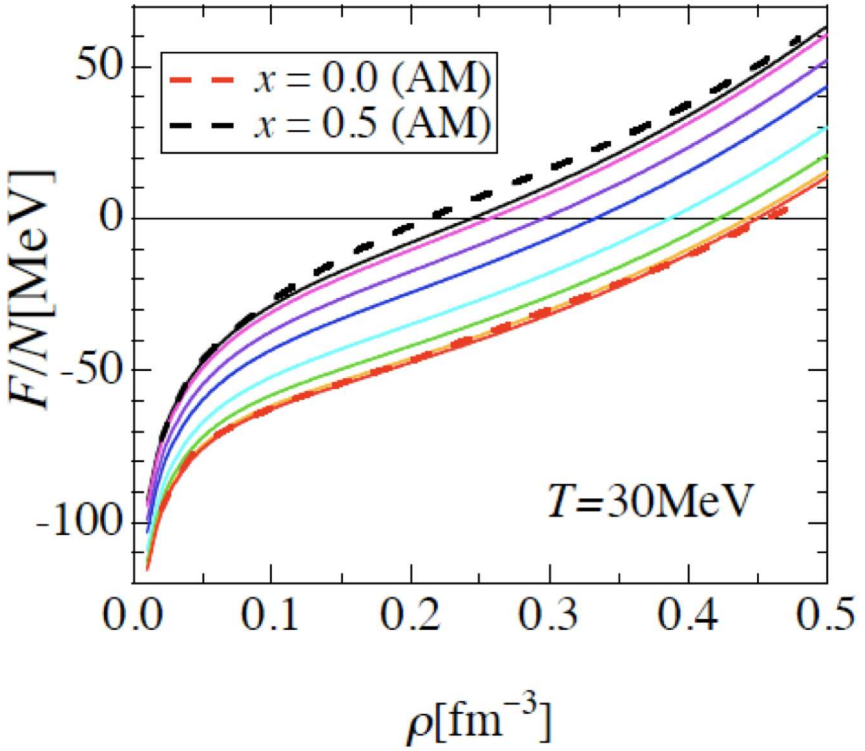
← Effective mass

$$\frac{S_0}{N} = -\frac{k_B}{N}\sum_{i=\mathrm{p,n}}\sum_j \left\{\left[1 - n_i(k_j)\right]\ln\left[1 - n_i(k_j)\right] + n_i(k_j)\ln\left[n_i(k_j)\right]\right\}$$

Free energy is minimized with respect to m_p^* and m_n^*

Free energy of asymmetric nuclear matter





H. Togashi et al., Nucl. Phys. A902 (2013) 53.

Equation of state for non-uniform nuclear matter

*We use the **Thomas-Fermi method** by Shen et al.*

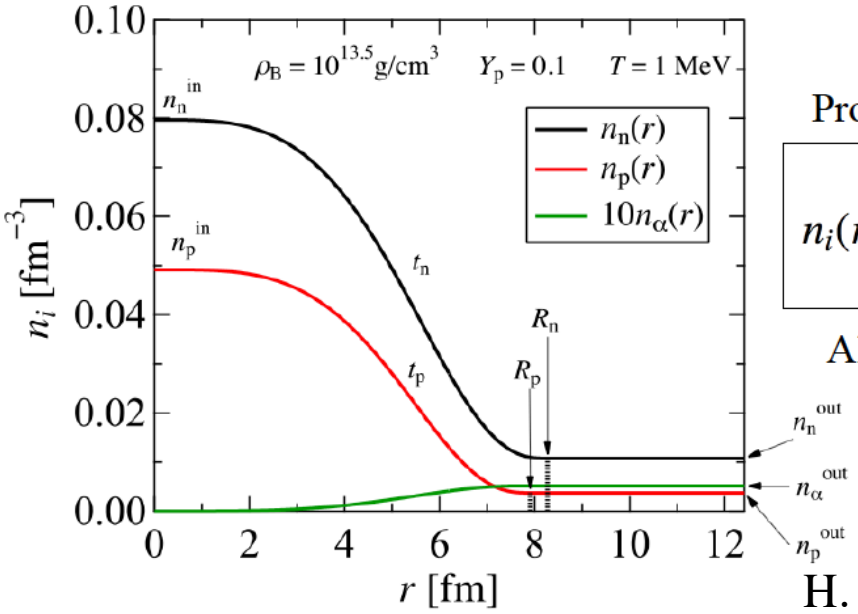
(PTP 100 (1998) 1013, APJS 197(2011) 20)

Free energy of a Wigner-Seitz cell

$$F = \int d\boldsymbol{r} f(n_{\rm p}(r), n_{\rm n}(r), n_\alpha(r)) + F_0 \int d\boldsymbol{r} |\nabla(n_{\rm p}(r) + n_{\rm n}(r))|^2 + \frac{e^2}{2} \int d\boldsymbol{r} \int d\boldsymbol{r}' \frac{[n_{\rm p}(r) + 2n_\alpha(r) - n_{\rm e}][n_{\rm p}(r') + 2n_\alpha(r') - n_{\rm e}]}{|\boldsymbol{r} - \boldsymbol{r}'|} + c_{\rm bcc} \frac{(Ze)^2}{a}$$

Bulk energy — **Gradient energy** — **Coulomb energy**

Free energy density of uniform matter: $f = f_{\rm N} + f_\alpha$



Particle number density distributions

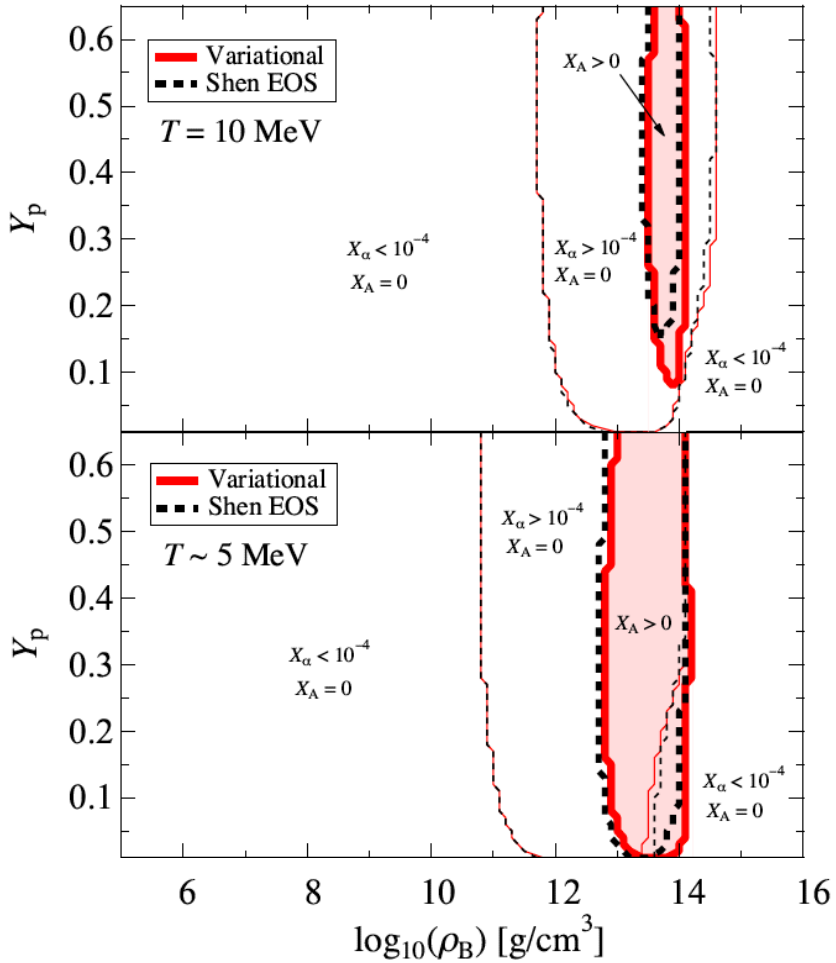
Protons and neutrons (i = p, n)

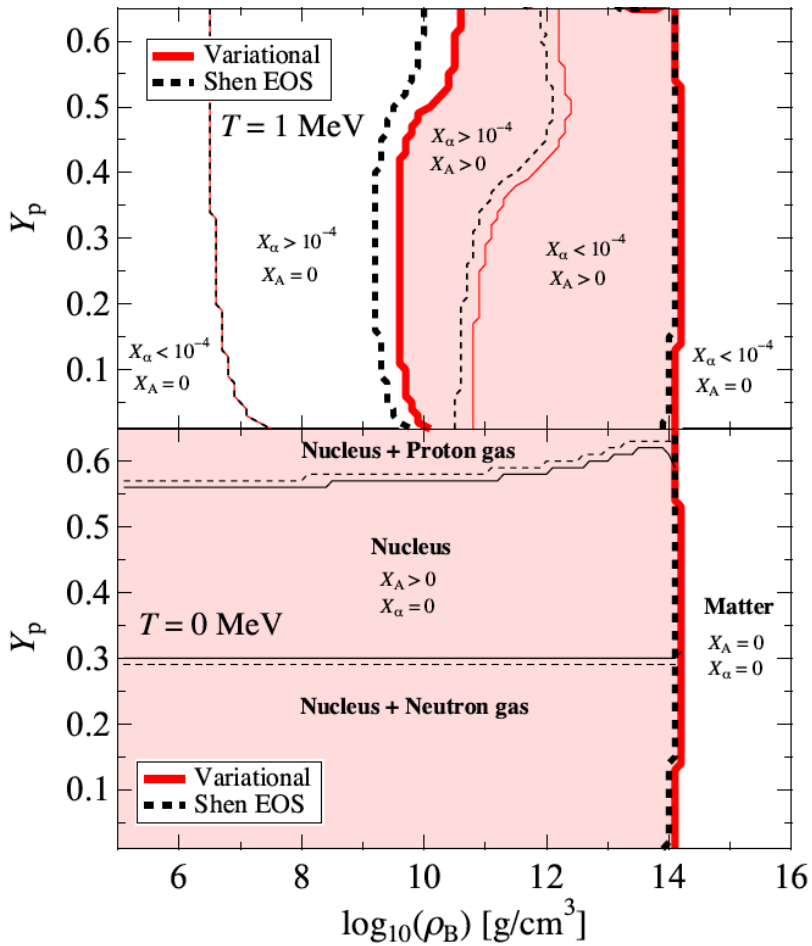
$$n_i(r) = \begin{cases} (n_i^{\rm in} - n_i^{\rm out})[1 - (r/R_i)^{t_i}]^3 + n_i^{\rm out} & (0 \le r \le R_i) \\ n_i^{\rm out} & (R_i \le r \le R_{\rm cell}) \end{cases}$$

Alpha-particles

$$n_\alpha(r) = \begin{cases} -n_\alpha^{\rm out}[1 - (r/R_{\rm p})^{t_{\rm p}}]^3 + n_\alpha^{\rm out} & (0 \le r \le R_{\rm p}) \\ n_\alpha^{\rm out} & (R_{\rm p} \le r \le R_{\rm cell}) \end{cases}$$

Phase Diagram of Nuclear Matter





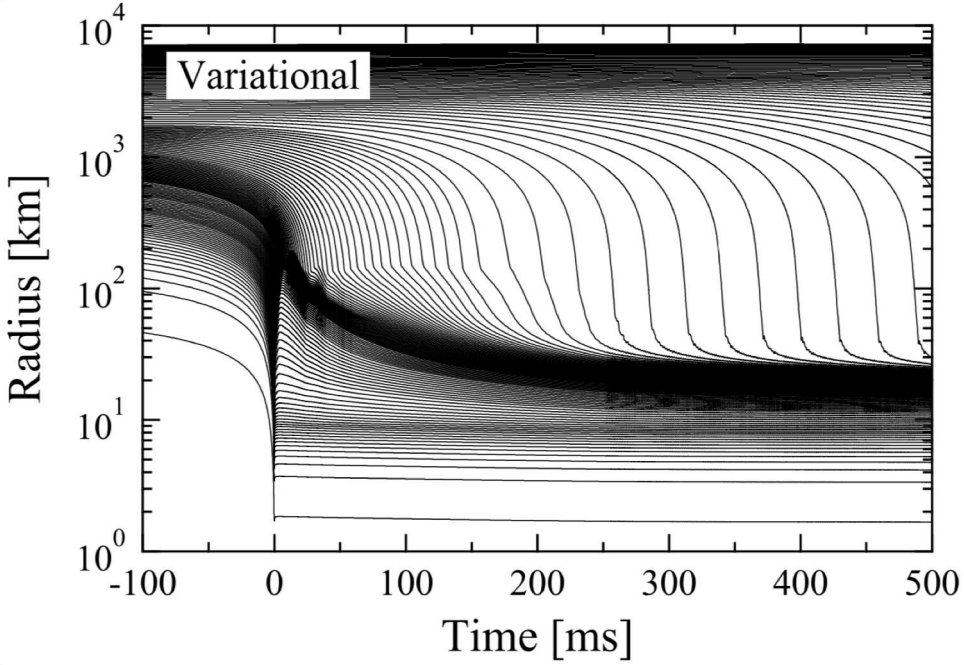
H. Togashi et al., Nucl. Phys. A961 (2017)78

Application to Core-Collapse Supernovae

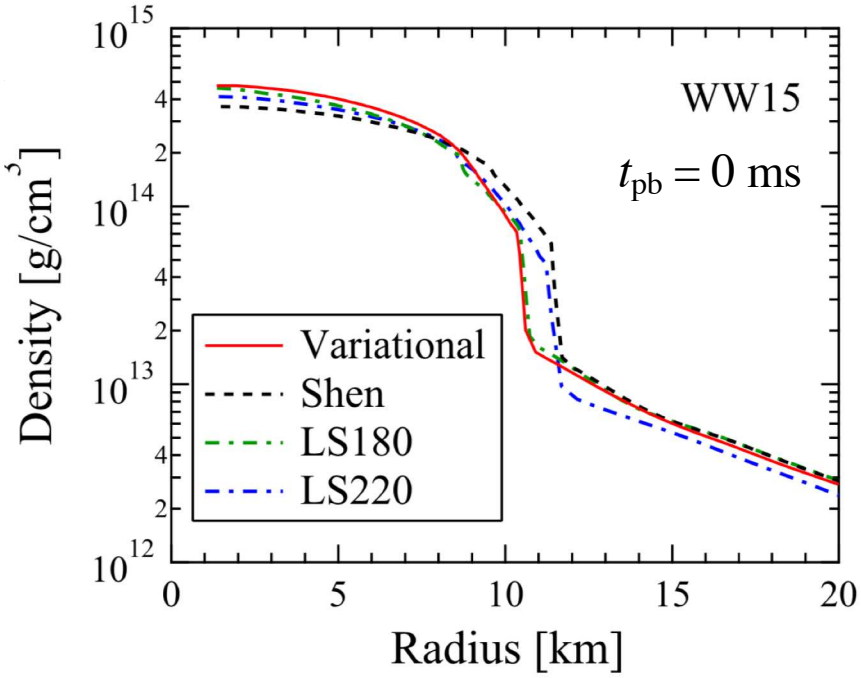
1D neutrino-radiation hydrodynamics simulations

Progenitor: Woosley Weaver 1995, $15M_\odot$ (K. Sumiyoshi, et al., NPA 730 (2004) 227) (Astrophys. J. Suppl. 101 (1995) 181)

More sophisticated multi-dimensional simulation
→ Takiwaki-san's talk in this afternoon!



Radial trajectories of mass elements



Central density: 0.30 fm^{-3}
Temperature: ~10 MeV
Proton fraction: ~0.3

HT *et al.*, in preparation

Cluster Variational Method + NSE

IOP Publishing Journal of Physics G: Nuclear and Particle Physics

J. Phys. G: Nucl. Part. Phys. **44** (2017) 094001 (23pp) https://doi.org/10.1088/1361-6471/aa7f35

A new equation of state for core-collapse supernovae based on realistic nuclear forces and including a full nuclear ensemble*

S Furusawa[1,2,9], H Togashi[3,4], H Nagakura[5], K Sumiyoshi[6], S Yamada[7], H Suzuki[8] and M Takano[4,7]

Cluster Variational Method + NSE

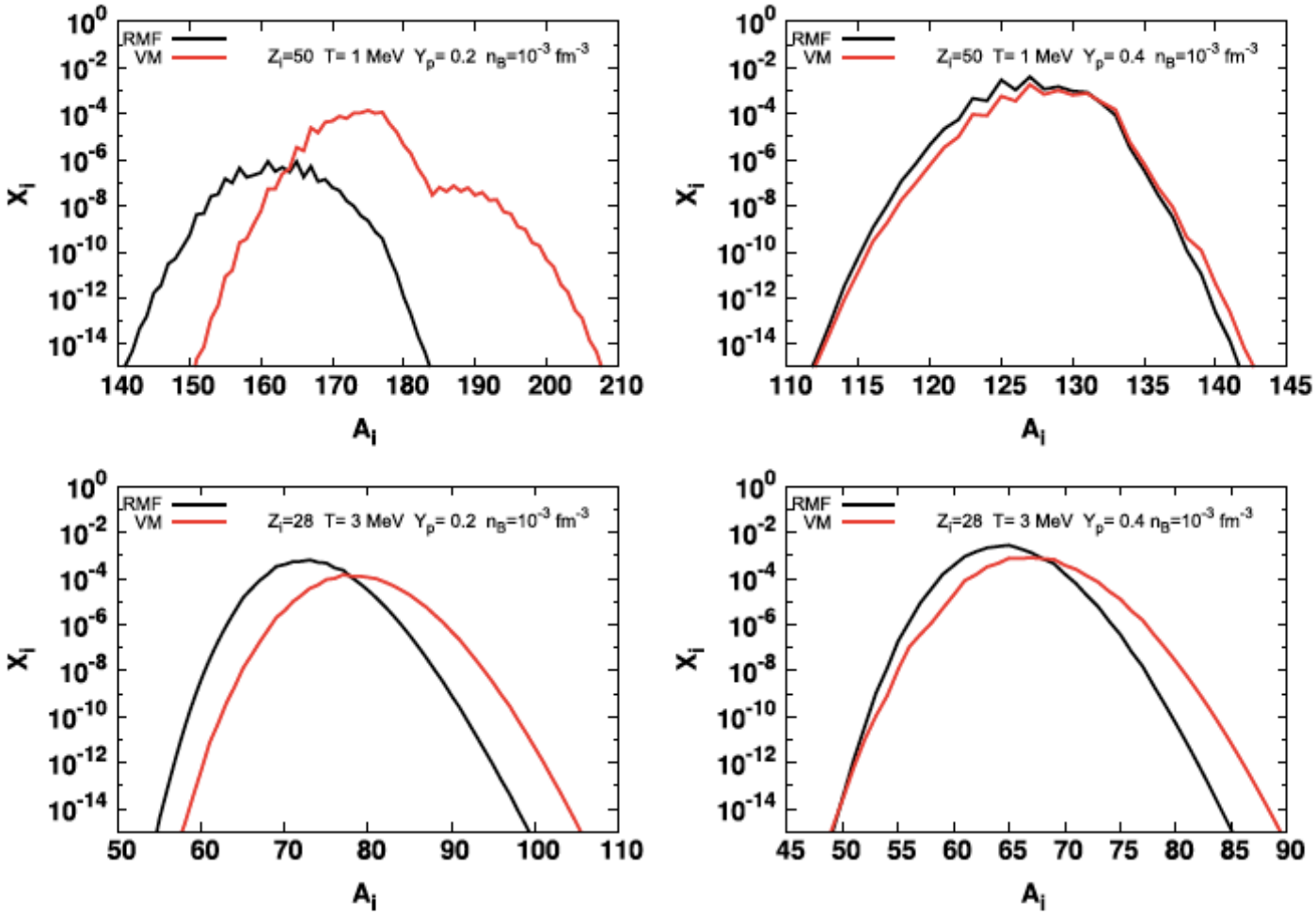


Figure 5. Mass fractions of the isotopes of Sn ($Z_i = 50$) for $T = 1$ MeV (top row) and of Ni ($Z_i = 28$) for $T = 3$ MeV (bottom row) as a function of mass number for the new EOS (red lines) and the FYSS EOS (black lines) at $Y_p = 0.2$ (left column) and 0.4 (right column) and $n_B = 10^{-3}$ fm^{-3}.

III. Variational Method with Explicit Energy Functional

K. Shoji, R. Yokota, H. Togashi, MT

Hamiltonian

$$H = -\sum_{i=1}^{N} \frac{\hbar^2}{2m} \nabla_i^2 + \sum_{i<j}^{N} \boldsymbol{V_{ij}} + \sum_{i<j<k}^{N} V_{ijk}$$

N : The number of nucleons
m : The mass of nucleons

The AV6' potential [1]

$$\boldsymbol{V_{ij}} = \sum_{t=0}^{1} \sum_{s=0}^{1} V_{ts}(r_{ij})\, P_{tsij} + \sum_{t=0}^{1} V_{\mathrm{T}t}(r_{ij}) S_{\mathrm{T}ij} P_{t1ij}$$

s, t : total spin (isospin) of two nucleons
P_{tsij} : spin-isospin projection operator

$$S_{\mathrm{T}ij} = 3\frac{(\boldsymbol{\sigma}_i \cdot \boldsymbol{r}_{ij})(\boldsymbol{\sigma}_j \cdot \boldsymbol{r}_{ij})}{r_{ij}^2} - (\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j)$$

: tensor operator

[1] R. B. Wiringa and S. C. Pieper, Phys. Rev. Lett. 89 (2002) 182501.

V_{ijk}: the UIX three-nucleon potential

Variational functions

Pure Neutron Matter (PNM) at $T = 0$

Spin-dependent radial distribution functions

ρ : the number density of nucleons

$$F_s(r) = \frac{1}{N\rho}\left\langle \sum_{i=1}^{N}\sum_{j=1}^{N} \delta^3(\boldsymbol{r} - \boldsymbol{r}_{ij}) P_{1sij} \right\rangle$$

Tensor distribution function

$$F_{\mathrm{T}}(r) = \frac{1}{N\rho}\left\langle \sum_{i=1}^{N}\sum_{j=1}^{N} \delta^3(\boldsymbol{r} - \boldsymbol{r}_{ij}) S_{\mathrm{T}ij} \right\rangle$$

Auxiliary functions

$$F_s(r) = F_{cs}(r) + 8s[g_{\mathrm{T}}(r)]^2 F_{s\mathrm{F}}(r)$$

$$F_{\mathrm{T}}(r) = 16\sqrt{F_{\mathrm{c1}}(r)F_{1\mathrm{F}}(r)}\, g_{\mathrm{T}}(r) - 16[g_{\mathrm{T}}(r)]^2 F_{1\mathrm{F}}(r)$$

$F_{cs}(r)$: Intrinsically-central distribution functions

$g_{\mathrm{T}}(r)$: Dressed tensor correlation function

Structure Functions

PNM at $T = 0$

$$S_{c1}(k) = \frac{1}{N}\left\langle \left| \sum_{i=1}^{N} \exp(i\boldsymbol{k}\cdot\boldsymbol{r_i}) \right|^2 \right\rangle = 1 + S_1(k) + S_0(k) \geq 0$$

$$S_{c2}(k) = \frac{1}{3N}\left\langle \left| \sum_{i=1}^{N} \boldsymbol{\sigma}_i \exp(i\boldsymbol{k}\cdot\boldsymbol{r_i}) \right|^2 \right\rangle = 1 + \frac{1}{3}S_1(k) - S_0(k) \geq 0$$

$$S_{cT1}(k) = \frac{1}{Nk^2}\left\langle \left| \sum_{i=1}^{N} (\boldsymbol{\sigma_i}\cdot\boldsymbol{k}) \exp(i\boldsymbol{k}\cdot\boldsymbol{r_i}) \right|^2 \right\rangle = S_{c2}(k) - \frac{S_T(k)}{3} \geq 0$$

$$S_{cT2}(k) = \frac{1}{2Nk^2}\left\langle \left| \sum_{i=1}^{N} (\boldsymbol{\sigma_i}\times\boldsymbol{k}) \exp(i\boldsymbol{k}\cdot\boldsymbol{r_i}) \right|^2 \right\rangle = S_{c2}(k) + \frac{S_T(k)}{6} \geq 0$$

$$S_S(k) = \rho \int [F_S(r) - F_S(\infty)]\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}} d\boldsymbol{r} \qquad S_T(k) = 4\pi\rho \int_0^\infty F_T(r) j_2(kr) r^2 dr$$

Energy functional

PNM at $T = 0$

$$\frac{E}{N} = \frac{E_1}{N} + \frac{\pi\hbar^2\rho}{2m}\sum_{s=0}^{1}\int_0^\infty F_{cs}(r)\left[\frac{1}{F_{cs}(r)}\frac{dF_{cs}(r)}{dr} - \frac{1}{F_{sF}(r)}\frac{dF_{sF}(r)}{dr}\right]^2 r^2 dr$$

$$+\frac{16\pi^2\hbar^2\rho}{m}\int_0^\infty\left[\left[\frac{dg_T(r)}{dr}\right]^2 + \frac{6}{r^2}[g_T(r)]^2\right]F_{1F}(r)r^2dr + \frac{E_{nod}}{N}$$

$$-\frac{\hbar^2}{16\pi^2 m\rho}\int_0^\infty \frac{[S_{c1}(k) + 2S_{cF}(k) - 3][S_{c1}(k) - S_c}{S_{c1}(k)/S_{cF}(k)}$$

$$-\frac{\hbar^2}{16\pi^2 m\rho}\sum_{n=1}^{2}\int_0^\infty n\frac{[S_{cTn}(k) + 2S_{cF}(k) - 3][S_{cTn}(k)}{S_{cTn}(k)/S_{cF}(k)}$$

$$+2\pi\rho\int_0^\infty\left[\sum_{s=0}^{1}F_s(r)V_{1s}(r) + F_T(r)V_{T1}(r)\right]r^2dr + \frac{E^R}{N} + \frac{E^{2\pi}}{N}$$

Energy functional

PNM at $T = 0$

Caused by the correlation

$$\frac{E}{N} = \frac{E_1}{N} + \frac{\pi\hbar^2\rho}{2m}\sum_{s=0}^{1}\int_0^\infty F_{cs}(r)\left[\frac{1}{F_{cs}(r)}\frac{dF_{cs}(r)}{dr} - \frac{1}{F_{sF}(r)}\frac{dF_{sF}(r)}{dr}\right]^2 r^2 dr$$

1-Body

Kinetic energy

$$+\frac{16\pi^2\hbar^2\rho}{m}\int_0^\infty \left[\left[\frac{dg_T(r)}{dr}\right]^2 + \frac{6}{r^2}[g_T(r)]^2\right] F_{1F}(r) r^2 dr + \frac{E_{nod}}{N}$$

$$-\frac{\hbar^2}{16\pi^2 m\rho}\int_0^\infty \frac{[S_{c1}(k) + 2S_{cF}(k) - 3][S_{c1}(k) - S_c}{S_{c1}(k)/S_{cF}(k)}$$

$$-\frac{\hbar^2}{16\pi^2 m\rho}\sum_{n=1}^{2}\int_0^\infty n\frac{[S_{cTn}(k) + 2S_{cF}(k) - 3][S_{cTn}(k)}{S_{cTn}(k)/S_{cF}(k)}$$

potential energy

$$+2\pi\rho\int_0^\infty \left[\sum_{s=0}^{1} F_s(r)V_{1s}(r) + F_T(r)V_{T1}(r)\right] r^2 dr + \frac{E^R}{N} + \frac{E^{2\pi}}{N}$$

2BF

3BF

Energy functional

PNM at $T = 0$

kinetic energy

Jastrow-type wave function

$$\Psi(x_1,\cdots,x_N) = \mathrm{Sym}\left[\prod_{i<j}^{N} f_{ij}\right]\Phi(x_1,\cdots,x_N)$$

symmetrizer

correlation function

The w.f. for the Fermi gas

cluster expansion of

$$\left\langle\Psi\left|-\sum\frac{\hbar^2}{2m}\nabla_i^2\right|\Psi\right\rangle$$

Our energy expression includes …

2-body cluster term : completely

3-body cluster term : the lowest order { direct terms; 2−nucleon exchange terms

4-body cluster term : the lowest order direct terms

Energy functional

PNM at $T = 0$

potential energy

- Two-body potential … **exact expression**
- Three-body potential

$$V_{ijk} = V_{ijk}^{R} + V_{ijk}^{2\pi}$$

The repulsive part

$$\frac{E^{\mathrm{R}}}{N} = \frac{\rho^2}{6} \int F_3(\boldsymbol{r}_i, \boldsymbol{r}_j, \boldsymbol{r}_k) V_{ijk}^{R} d\boldsymbol{r}_{ij} d\boldsymbol{r}_{jk}$$

the extended Kirkwood's assumption

$$F_3(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3) = F(r_{12}) F(r_{23}) F(r_{31}) \frac{F_{3\mathrm{F}}}{F_{\mathrm{F}}(r_{12}) F_{\mathrm{F}}(r_{23}) F_{\mathrm{F}}(r_{31})}$$

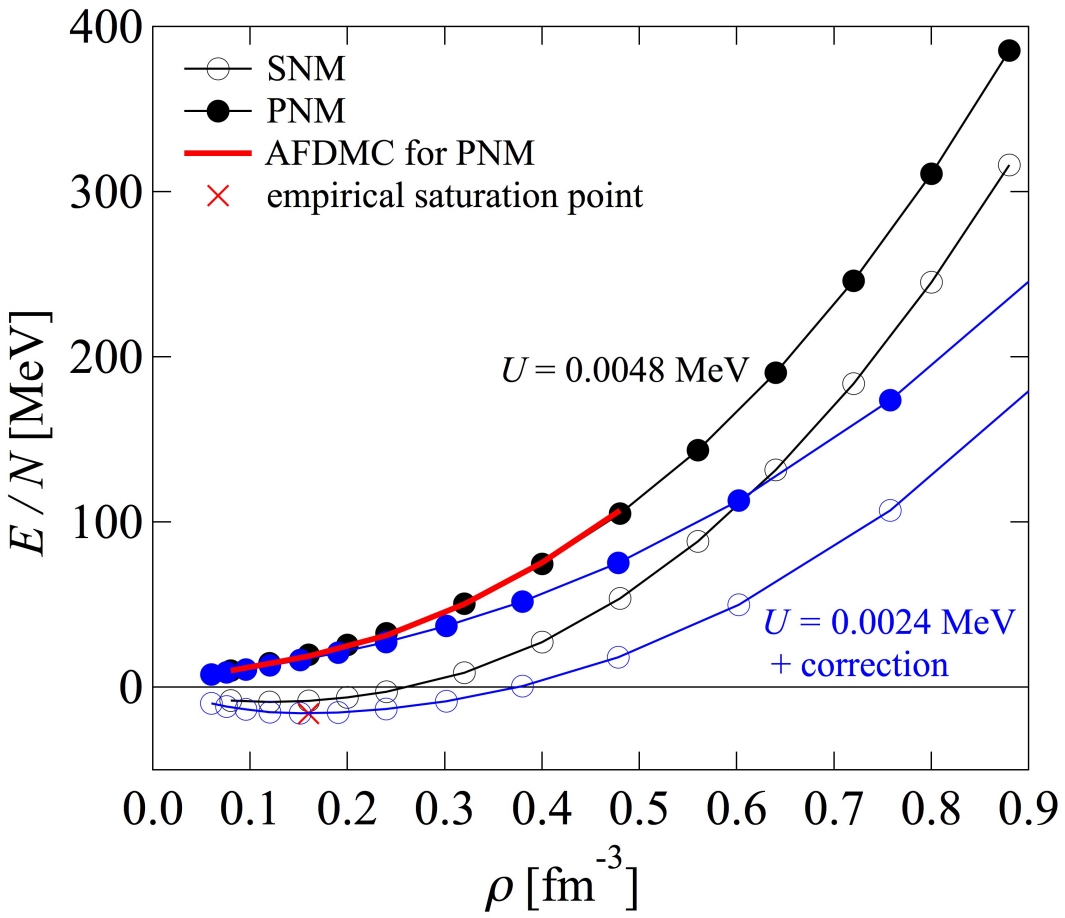
$$F(r) = \sum_{s=0}^{1} F_s(r)$$

The two-pion exchange part

Cluster expansion of $\langle \Psi | V_{ijk}^{2\pi} | \Psi \rangle$

The lowest order direct
three-body cluster term
+
Kirkwood's assumption

EOSs for cold nuclear matter



$U = 0.0048$ MeV

$U = \mathbf{0.0024}$ MeV

+

correction for SNM : $+\boldsymbol{\gamma_2 \rho^2 e^{-\gamma_3 \rho}}$

$(\gamma_2 = -1410\text{MeV fm}^6, \gamma_3 = 14.17\text{fm}^3)$

	calculated	empirical
$\rho_{\text{sat}}[\text{fm}^{-3}]$	0.16	0.16
$E_{\text{sat}}[\text{MeV}]$	-15.8	-15.8
$J[\text{MeV}]$	33.28	31.6 ± 2.7
$L[\text{MeV}]$	53.51	58.9 ± 16
$K[\text{MeV}]$	235.9	230 ± 20

[1] A. Lovato, O. Benhar, S. Fantoni, A. Yu. Illarionov and K. E. Schmidt, Phys. Rev. C83 (2011) 054003.

Variational method at finate temperature

following the method by Schmidt and Pandharipande [1]

$$\frac{F_0}{N} = \frac{E_{0\mathrm{T}}}{N}[F_s(r,T), F_{\mathrm{T}}(r,T), n(k;m^*,T)] - T\frac{S_0}{N}[n(k;m^*,T)]$$

Approximate internal energy

$n(k;m,T)$: The average occupation probabilities of single nucleon state

$$\frac{E}{N}[F_s(r), F_{\mathrm{T}}(r), n(k;m,T=0)] \Rightarrow \frac{E_{0\mathrm{T}}}{N}[F_s(r,T), F_{\mathrm{T}}(r,T), n(k;m^*,T>0)]$$

$$n(k;m,0) = \begin{cases} 0 & (k > k_{\mathrm{F}}) \\ 1 & (k \le k_{\mathrm{F}}) \end{cases}$$

$$n(k;m^*,T) = \left\{1 + \exp\left[\frac{\varepsilon(k)-\mu_0}{k_{\mathrm{B}}T}\right]\right\}^{-1} \quad \varepsilon(k) = \frac{\hbar^2 k^2}{2m^*}$$

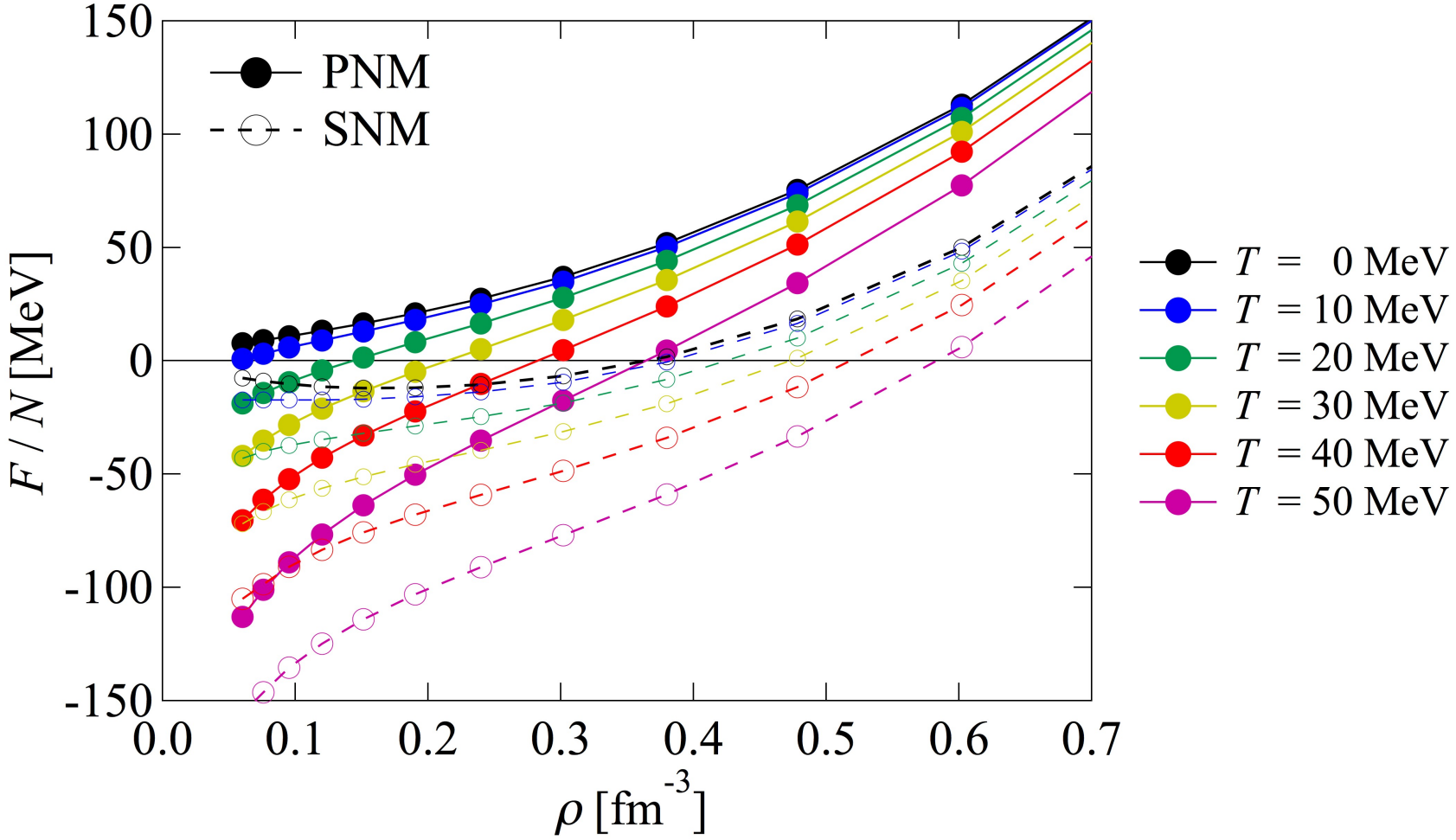
$$F_s(r) = \frac{1}{N\rho}\left\langle \sum_{i=1}^{N}\sum_{j=1}^{N} \delta^3(\boldsymbol{r}-\boldsymbol{r}_{ij}) P_{1sij} \right\rangle \qquad F_s(r,T) = \frac{1}{N\rho}\left\langle \sum_{i=1}^{N}\sum_{j=1}^{N} \delta^3(\boldsymbol{r}-\boldsymbol{r}_{ij}) P_{1sij} \right\rangle_{T}$$

Approximate entropy

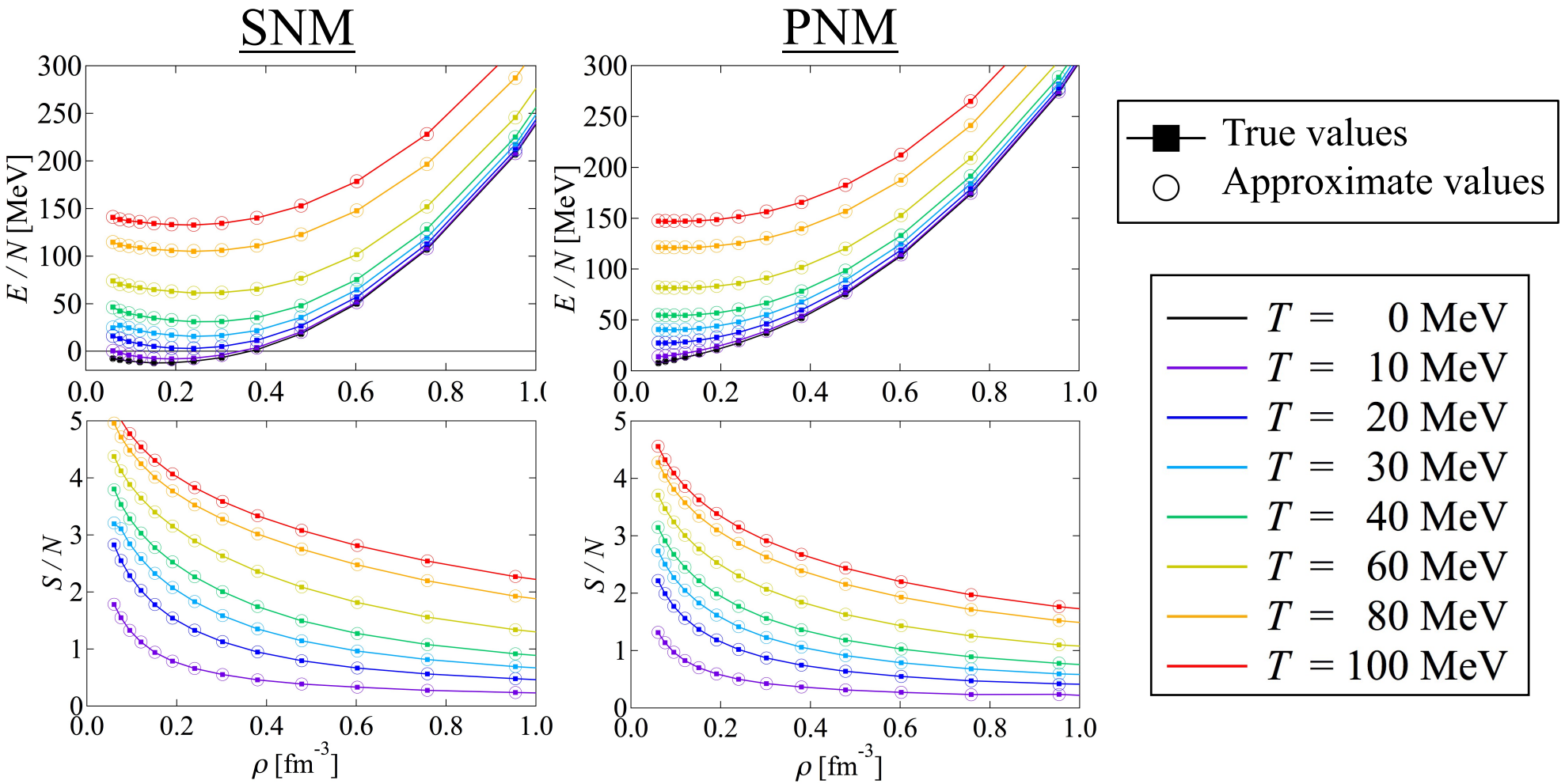
$$\frac{S_0}{N}[n(k;m^*,T)] = -\frac{k_B}{\pi\rho}\int_0^\infty \{[1-n]\ln[1-n] + n\ln n\}k^2 dk$$

[1] K. E. Schmidt and V. R. Pandharipande, Phys. Lett. **87B** (1979) 11.

EOSs for hot nuclear matter



EOSs for hot nuclear matter



Isovector spin-longitudinal (IVSL) response [1]

Sum of IVSL response

$$S_{\mathrm{L}}(k,T) = \frac{1}{Nk^2}\int_0^\infty R_{\mathrm{L}}(k,\omega,T)\,d\omega$$

Energy weighted sum of IVSL response

$$W_{\mathrm{L}}(k,T) = \frac{1}{Nk^2}\int_0^\infty R_L(k,\omega,T)\omega\,d\omega$$

Mean energy of IVSL response

$$E_{\mathrm{L}}(k,T) = \frac{W_{\mathrm{L}}(k,T)}{S_{\mathrm{L}}(k,T)}$$

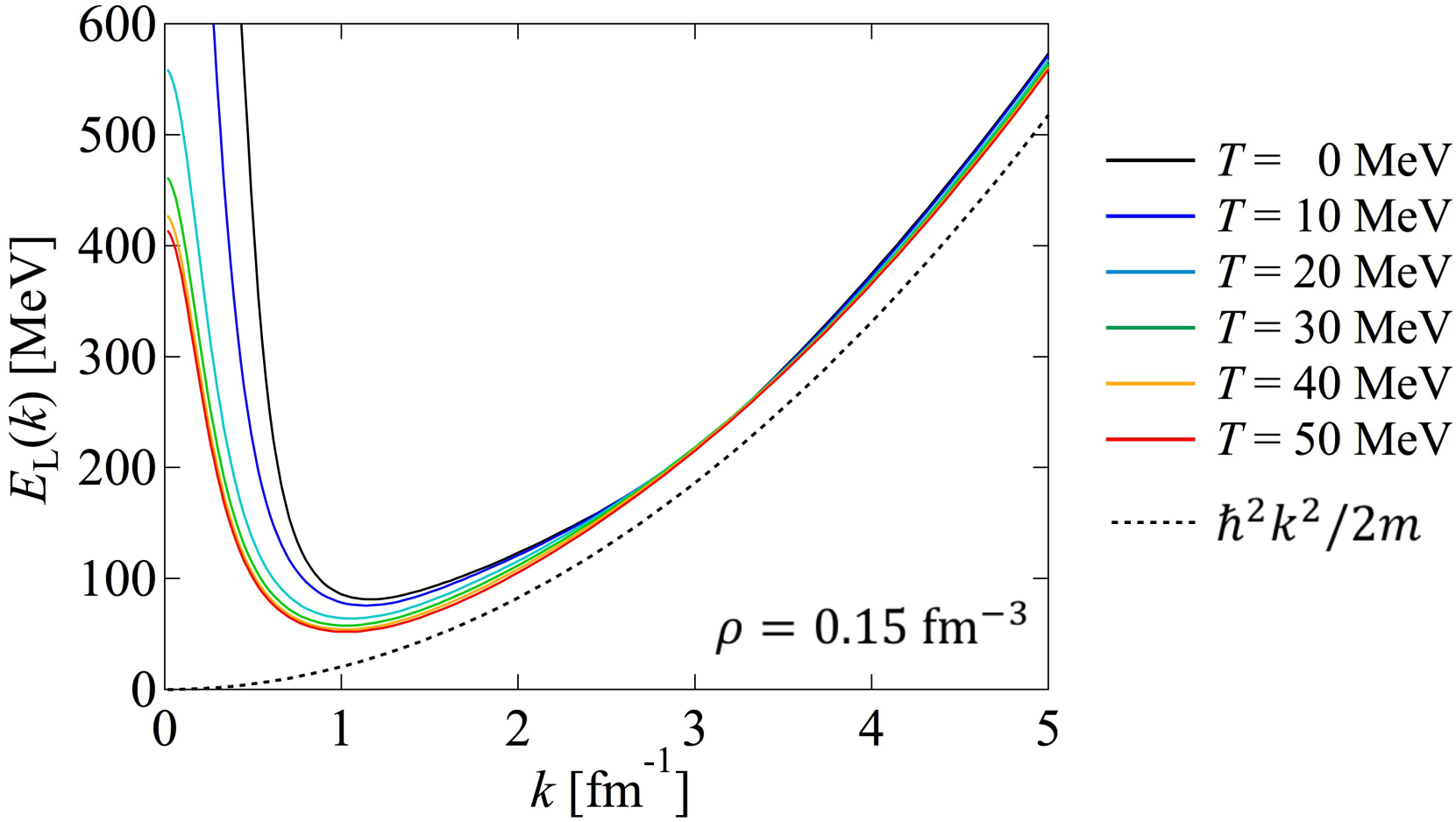
$$R_{\mathrm{L}}(k,\omega,T) = \sum_I\sum_J p_J(T)|\langle I|O_{\mathrm{L}}(\boldsymbol{k})|J\rangle|^2\delta(\omega_I-\omega_J-\omega)$$

$$O_L(\boldsymbol{k}) = \sum_i \boldsymbol{\sigma}_i\cdot\boldsymbol{k}\,\boldsymbol{\tau}_i\cdot\hat{\boldsymbol{t}}\exp(i\boldsymbol{k}\cdot\boldsymbol{r}_i)$$

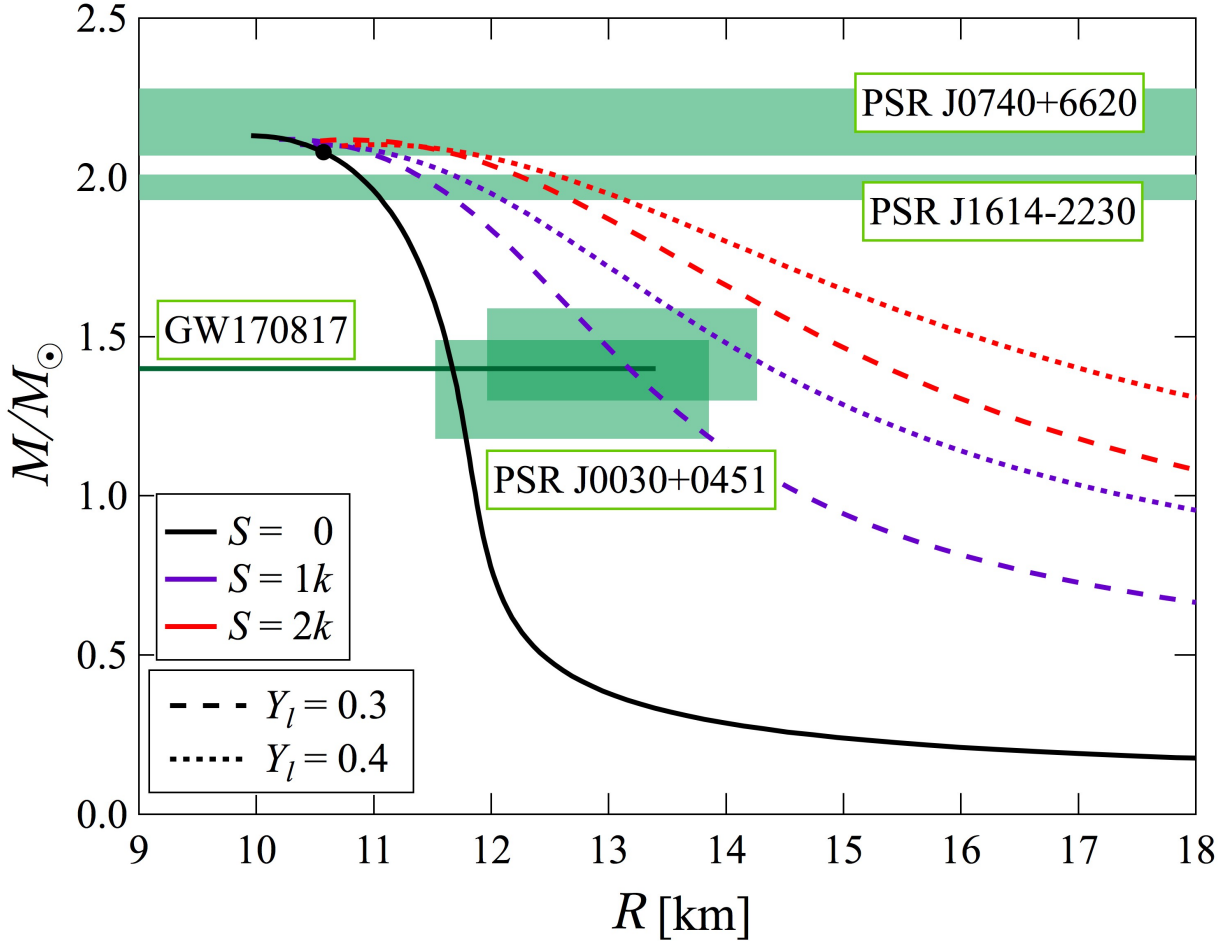
ω_I : the energy of the state $|I\rangle$
$p_J(T)$: the probability of $|I\rangle$ occurring in the thermal ensemble.
$\hat{\boldsymbol{t}}$: a unit vector in isospin space

[1] A. Mukherjee, Phys. Rev. **C79** (2009) 045811.

Mean energy of IVSL response for PNM



Application to proto-neutron stars



- EOS for ***ANM*** is obtained by ***interpolation***
- ***static*** & ***spherically*** symmetric
- S and Y_l : ***constant***
- low density region : ***Togashi EOS*** [1]

Dimensionless tidal deformability

$\Lambda(1.4M_\odot) = 222$

Constraint from GW170817

$\Lambda(1.4M_\odot) = 190^{+390}_{-120}$

[1] H. Togashi, K. Nakazato, Y. Takehara, S. Yamamuro, H. Suzuki, and M. Takano, Nucl. Phys. **A 961** (2017) 78.

IV Modified Urca process with variational methods

A. Dehghan Niri et al., Phys. Rev. C93 (2016) 045806

$$n + n \rightarrow n + p + e + \bar{\nu}_e \qquad n + p + e \rightarrow n + n + \nu_e$$

Effective mass

$$Q_\nu^{\mathrm{Mn}(e)} = \frac{11513}{945} \frac{G^2\left(1 + 3c_A^2\right)}{c^4 \pi \hbar^{13} 2^{10}} m_n^{*3} m_p^* \frac{k_{F_p} k_{F_e}^2}{k_{F_n}^6} \mathcal{R}(k_{\mathrm{F}n})(k_{\mathrm{B}} T)^8$$

$$\mathcal{R}(k_{\mathrm{F}n}) = F_c^2 + 12.7 F_t^2$$

Central correlation $F_c = 4\pi k_{\mathrm{F}n}^3 \int_0^\infty dr\, r^2 j_0(k_{\mathrm{F}n} r)\left[f_c^{(nn)}(r) f_c^{(np)}(r) - 1\right]$

Tensor correlation $F_t = 4\pi k_{\mathrm{F}n}^3 \int_0^\infty dr\, r^2 j_2(k_{\mathrm{F}n} r) f_c^{(nn)}(r) f_t^{(np)}(r)$

Correlation functions：Lowest Order Constrained Variational(LOCV) method

n-n: $^1S_0$ central correlation n-p: $^3S_1$-$^3D_1$ central and tensor correlations

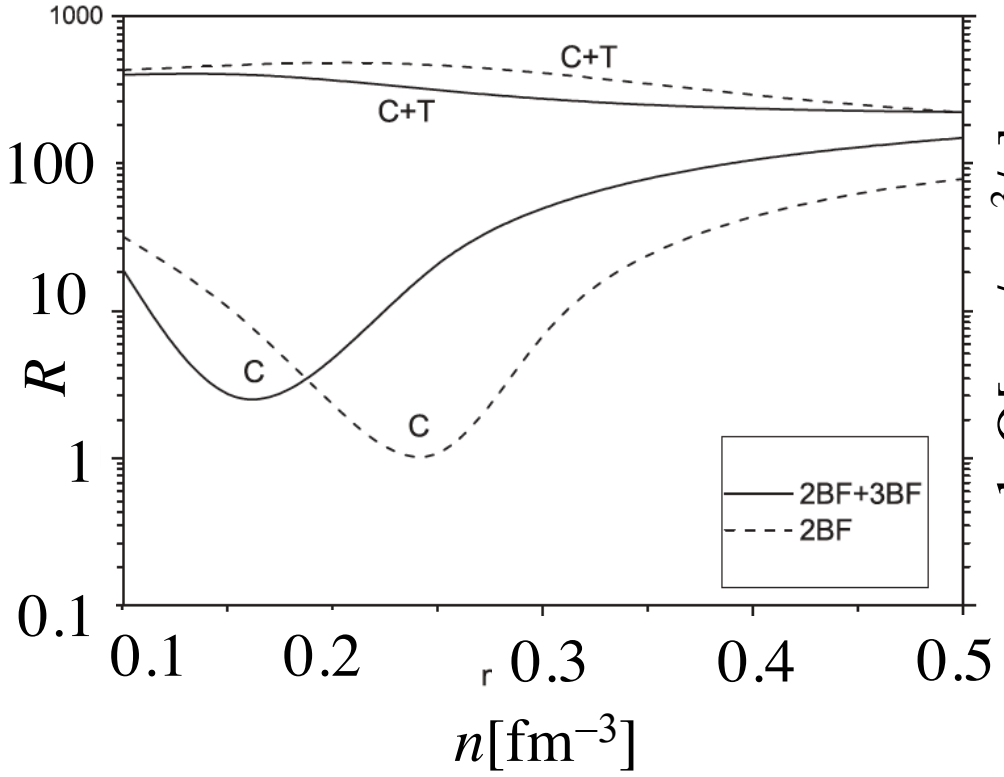
Effective mass：taken from a G matrix calculation

Modified Urca neutrino emissivity with LOCV

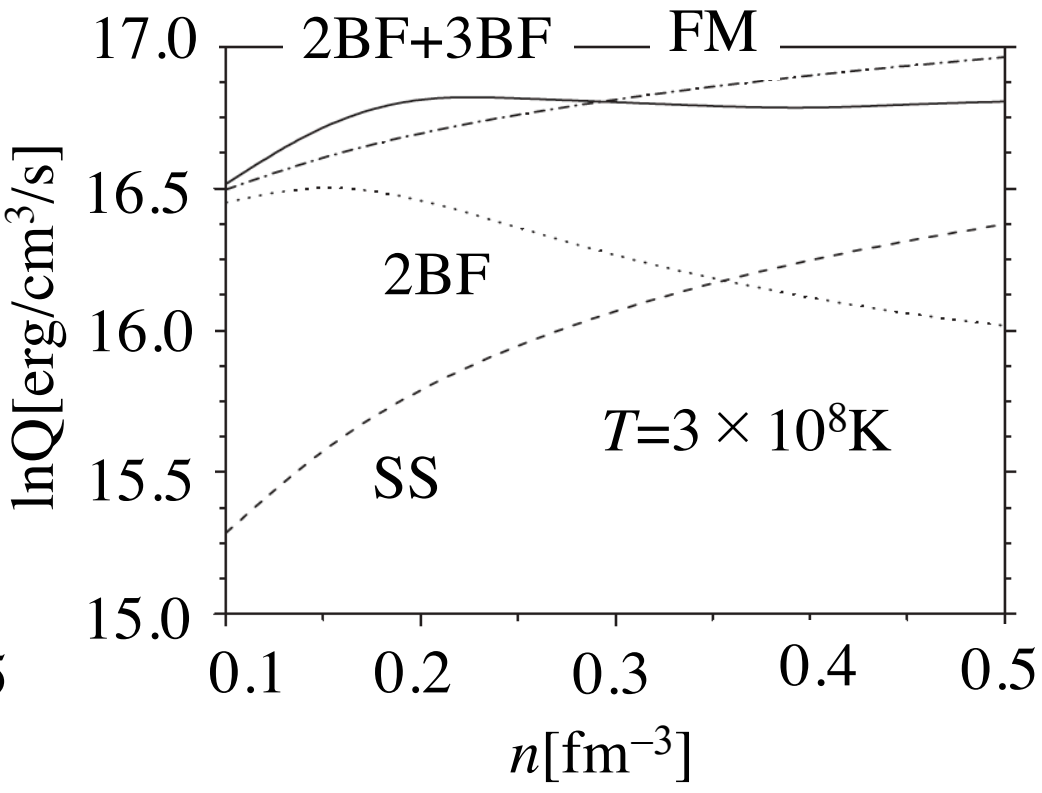
A. Dehghan Niri et al., Phys. Rev. C93 (2016) 045806

For neutron star matter

factor R



emissivity ln$Q$



FM: Friman and Maxwell, Astrophtys. J. 232(1979)541.

SS: R. F. Sawyer and A. Soni, Astrophys. J. 230 (1979)859.

Modified Urca neutrino emissivity with Togashi EOS

$$Q_\nu^{\mathrm{Mn}(e)} = \frac{11513}{945}\frac{G^2\left(1+3c_A^2\right)}{c^4\pi\hbar^{13}2^{10}} m_n^{*3} m_p^* \frac{k_{F_p}k_{F_e}^2}{k_{F_n}^6}\mathcal{R}(k_{\mathrm{F}n})(k_{\mathrm{B}}T)^8$$

$$(1+3c_{\mathrm{A}}^2)R(k_{\mathrm{Fn}}) = \left[F_{010}(k_{\mathrm{Fn}})\right]^2 + \frac{27}{2}\left[F_{\mathrm{T}11}(k_{\mathrm{Fn}})\right]^2 + c_{\mathrm{A}}^2\left\{3\left[F_{001}(k_{\mathrm{Fn}})\right]^2 + \frac{21}{2}\left[F_{\mathrm{T}00}(k_{\mathrm{Fn}})\right]^2 + 27\left[F_{\mathrm{T}11}(k_{\mathrm{Fn}})\right]^2\right\}$$

$$F_{sts'}(k) = 4\pi k^3\int_0^\infty \left[f_{\mathrm{C}1s}^-(r)f_{\mathrm{C}ts'}^0(r) - 1\right] j_0(kr)r^2dr$$

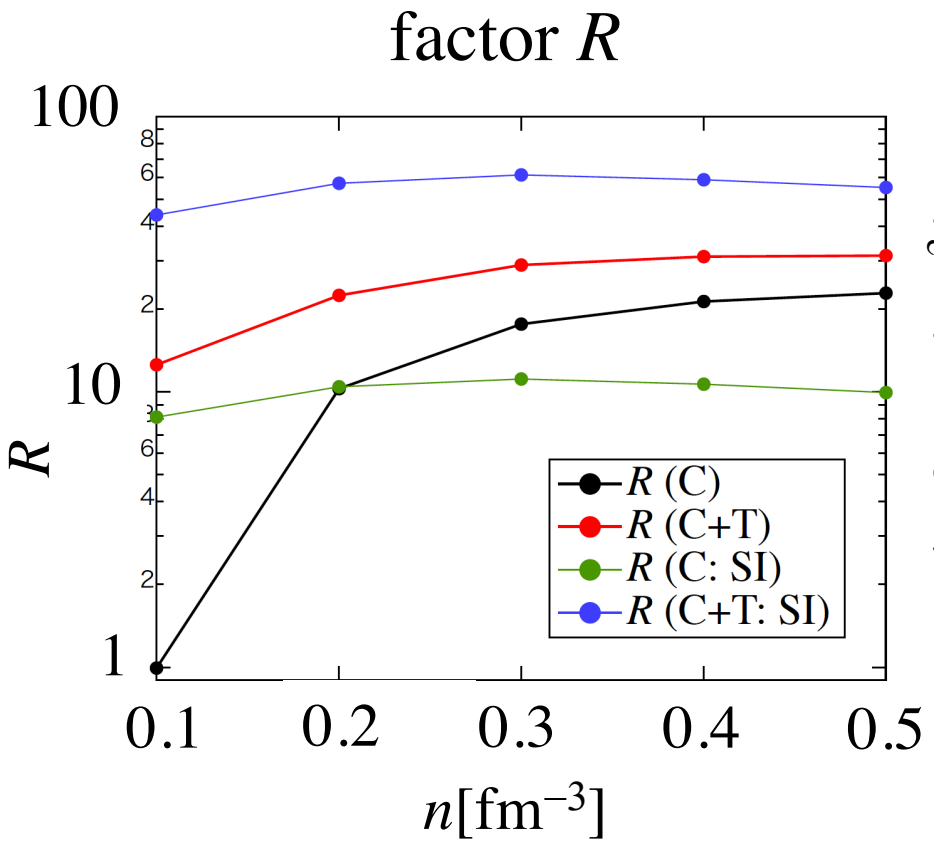
$$F_{\mathrm{T}st}(k) = 4\pi k^3\int_0^\infty f_{\mathrm{C}1s}^-(r)f_{\mathrm{T}t}^0(r) j_2(kr)r^2dr$$

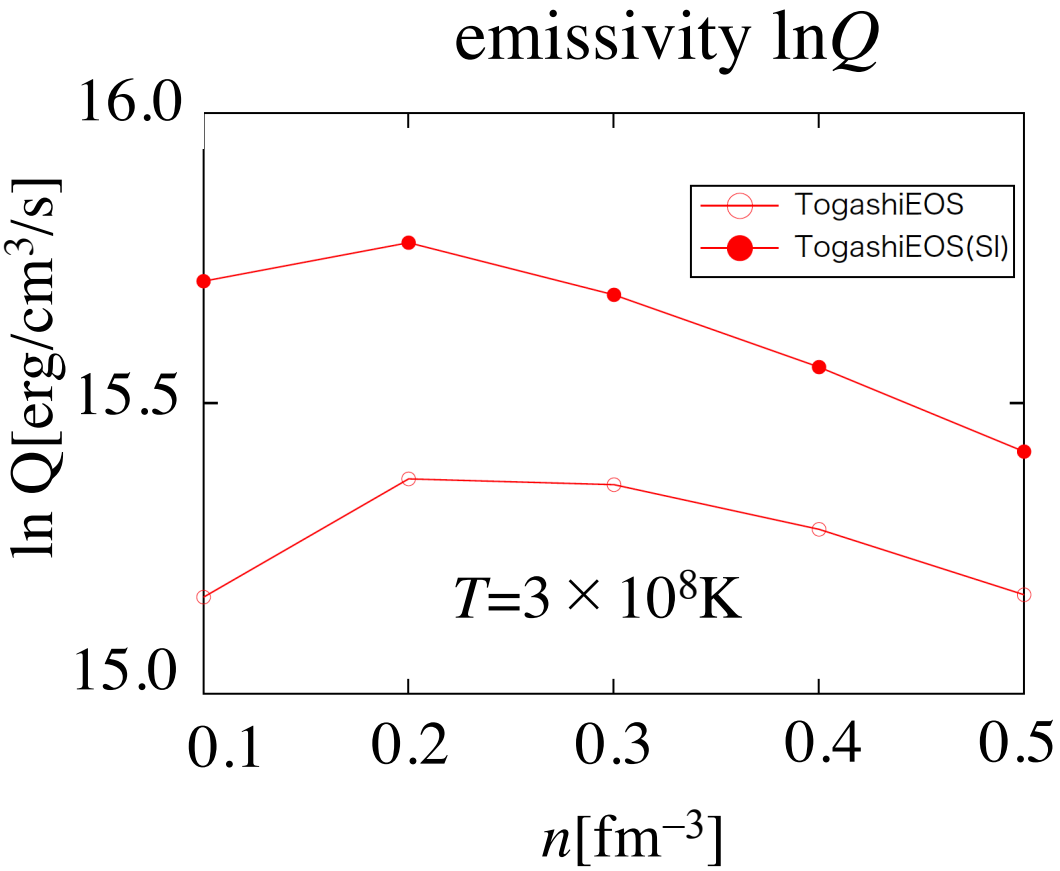
n-n: central correlation

n-p: central + tensor correlations

Modified Urca neutrino emissivity with Togashi EOS

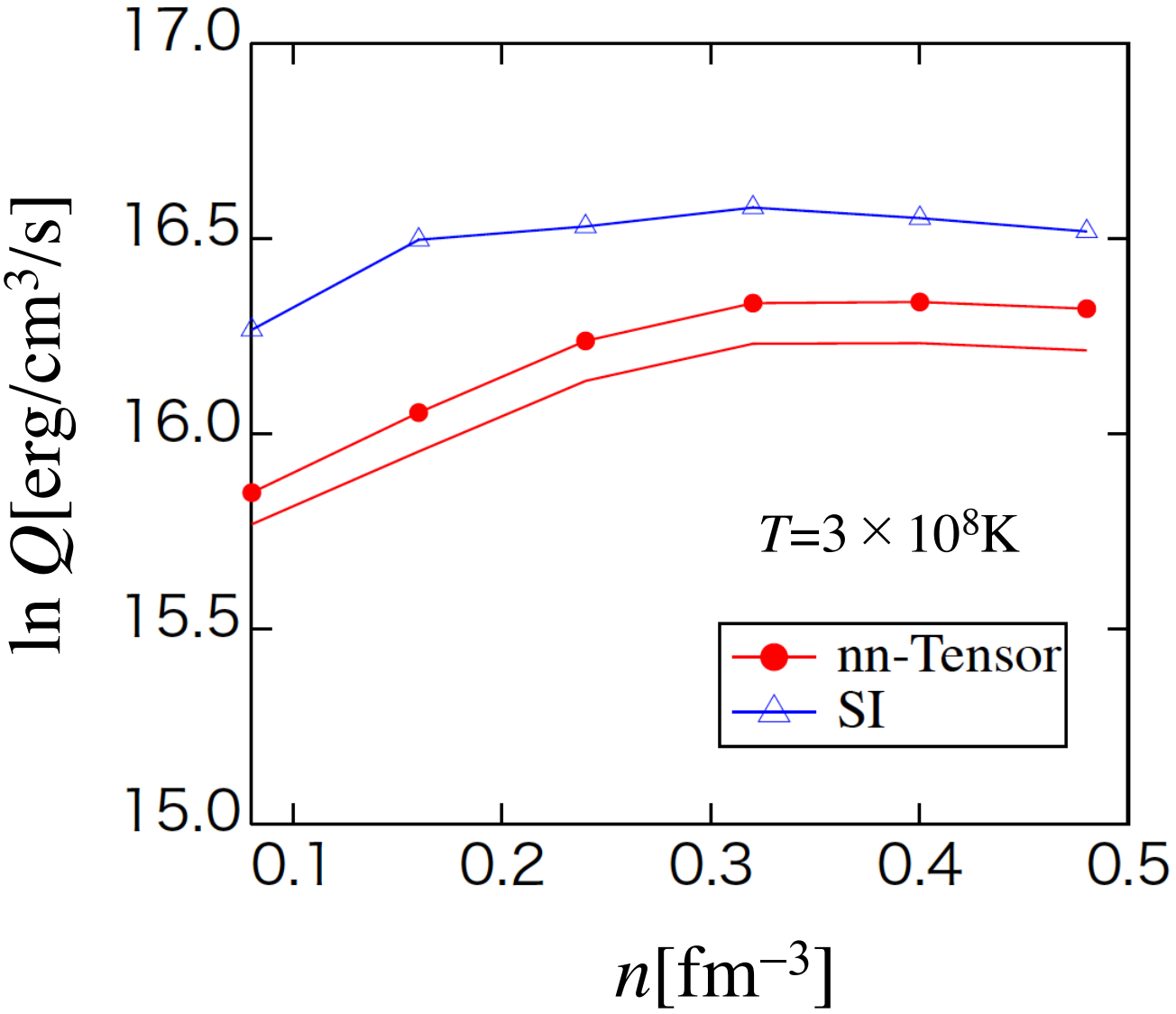
Neutron star matter　　Proton fraction: LOCV calculation





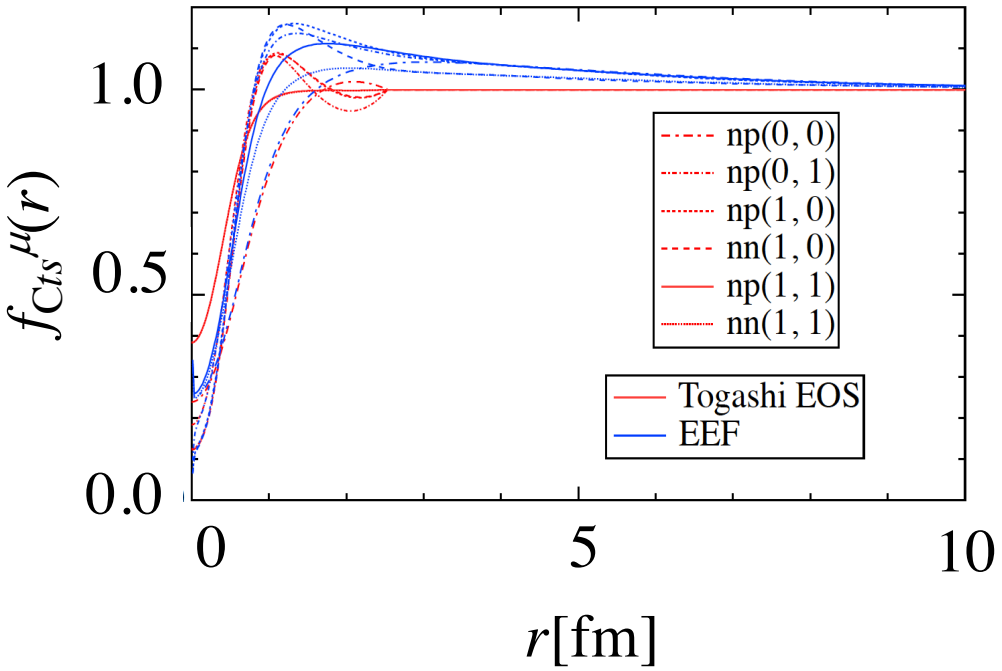
Modified Urca neutrino emissivity with the explicit energy functional

AV6'+UIX

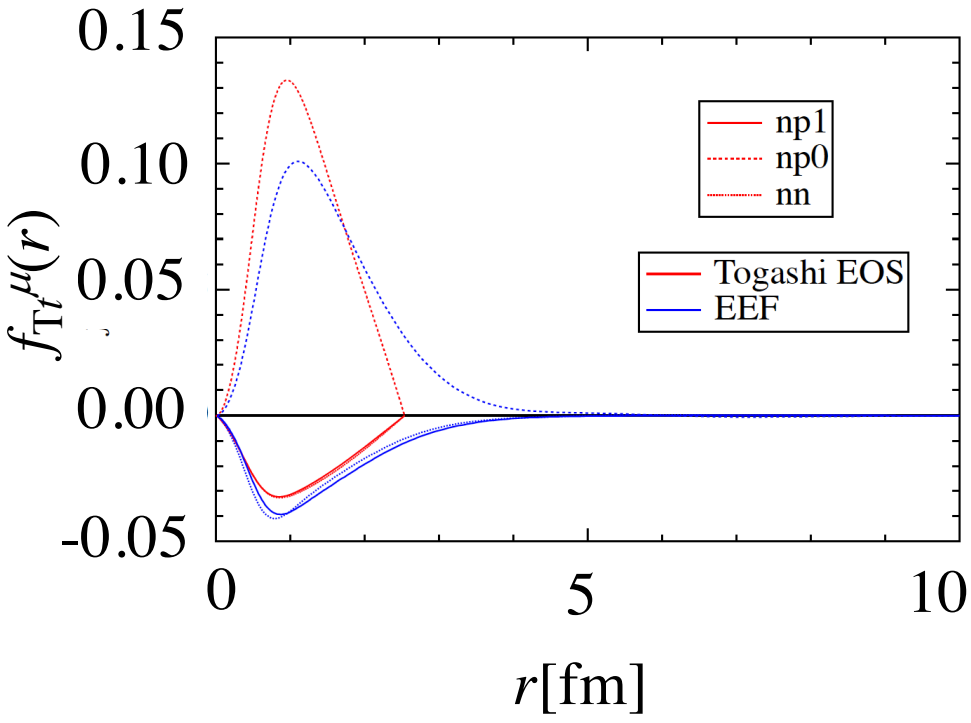


NN correlation functions

Central correlations



Tensor correlations



$n = 0.08\text{fm}^{-3}$

Central correlations with the EEF are in the two-body cluster approximation.

Modified Urca process and nucleon bremsstrahlung

Modified URCA process

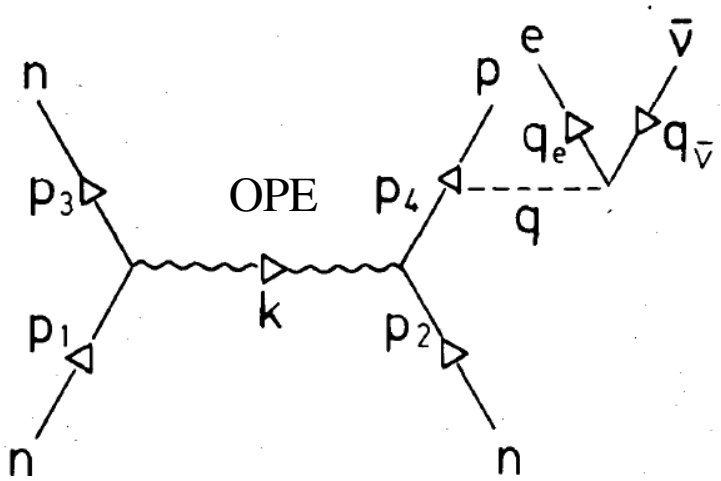
$$n + n \rightarrow n + p + e^- + \bar{\nu}_e$$

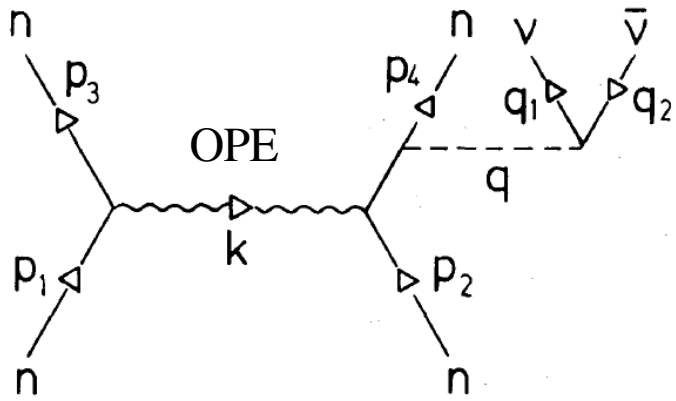
Nucleon bremsstrahlung

$$n + n \rightarrow n + n + \nu + \bar{\nu}$$

B. L. Friman and O. V. Maxwell, Astrophys. J. 232 (1979)451.

Perturbation theory：NN scattering by the one-pion exchange





$$\epsilon_{\mathrm{URCA}} = \frac{11513}{60480}\frac{G^2 g_A{}^2 m_n{}^{*3} m_p{}^*}{2\pi\hbar}\left(\frac{f}{m_\pi}\right)^4 p_{\mathrm{F}}(e)\alpha_{\mathrm{URCA}}(kT)^8$$

$$\epsilon_{nn} = \frac{41}{14175}\frac{G^2 g_A{}^2 m_n{}^{*4}}{2\pi\hbar}\left(\frac{f}{m_\pi}\right)^4 p_{\mathrm{F}}(n) F\left[\frac{m_\pi}{2p_{\mathrm{F}}(n)}\right](kT)^8$$

NN multiple scattering $\dfrac{1}{\omega^2} \rightarrow \dfrac{1}{\omega^2 + \Gamma^2/4}$

G. Raffelt and D. Seckel,
Phys. Rev. Lett. 67 (1991) 2605

Summary

Brief review of nuclear equations of state

Nuclear EOS based on the cluster variational method
for general purpose EOS: Togashi EOS

Variational method with explicit energy functionals:
Av6‘+UIX potentials for cold and hot nuclear matter.

First order transition to pion-condensed phase is not seen.

Applications to the modified Urca process

Tails of correlation functions

State dependence of correlation functions

play important roles.