General Relativistic Effects on Hill Stability of three-body systems

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2020 February 27th The Evolution of Massive Stars and Formation of Compact Stars: from the Cradle to the Grave

Contents

- **Introduction**
  - Relativistic multi-body systems
  - Hill Stability Problem
- **Numerical Simulation**
  - Method
  - Result
- **Theoretical Analysis**
  - Newtonian
  - post-Newtonian
- **Summary and Future Work**

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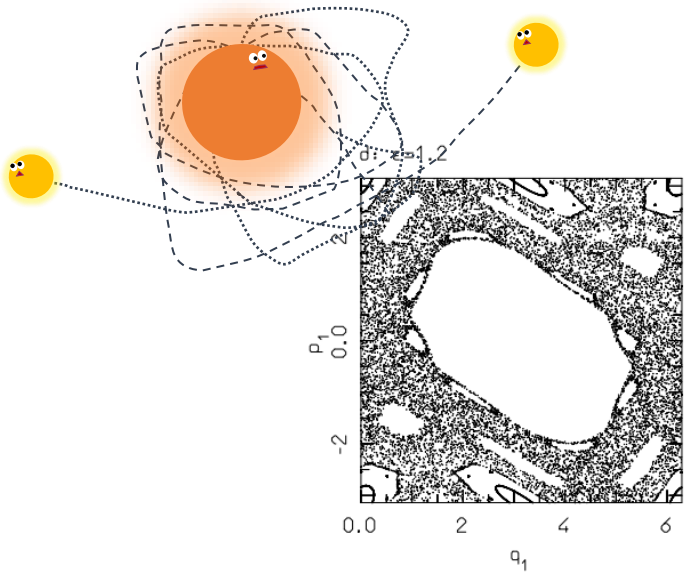
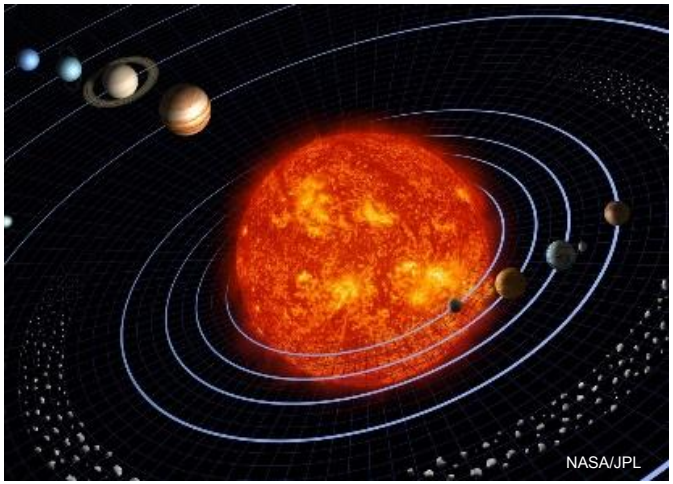
Introduction Relativistic multi-body systems

- **Dynamics of multi-body system**



Unstable state

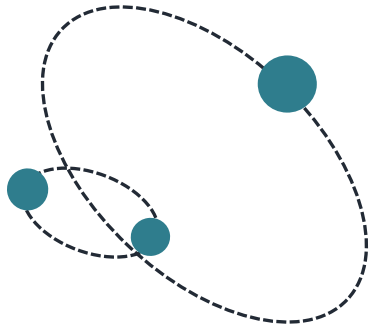


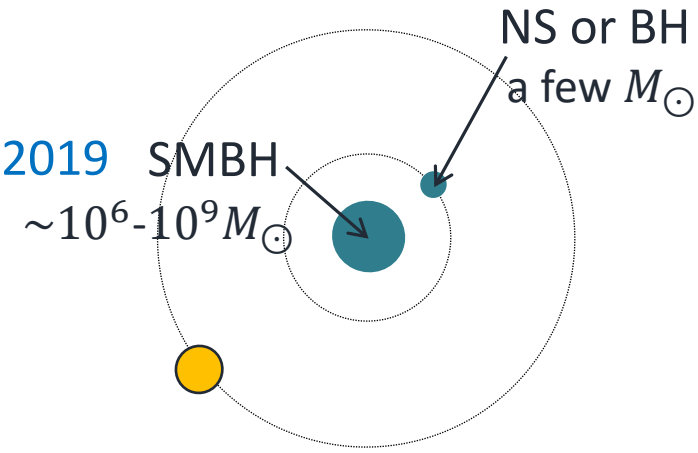
Stable state

- **Many studies in Newtonian mechanics**
  - Planetary systems
  - Formation of planets

Introduction Relativistic multi-body systems

- **Relativistic multi-body systems**
  - GW from hierarchical triple systems
    eg.) Randall and Xianyu 2019
  - Extreme Mass Ratio Inspiral (EMRI) with outer companion
    eg.) Amaro-Seoane et al. 2012, Bonga et al. 2019
  - N-body dynamics in galactic center or globular cluster
    eg.) Trani et al. 2019





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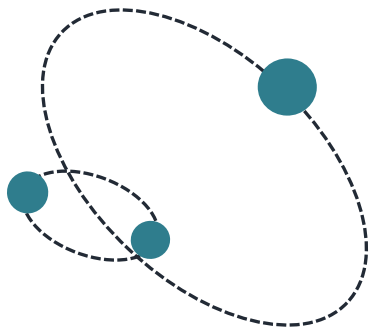
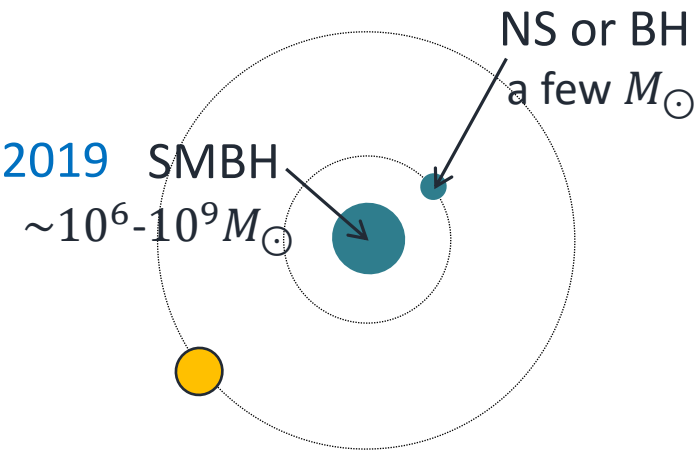
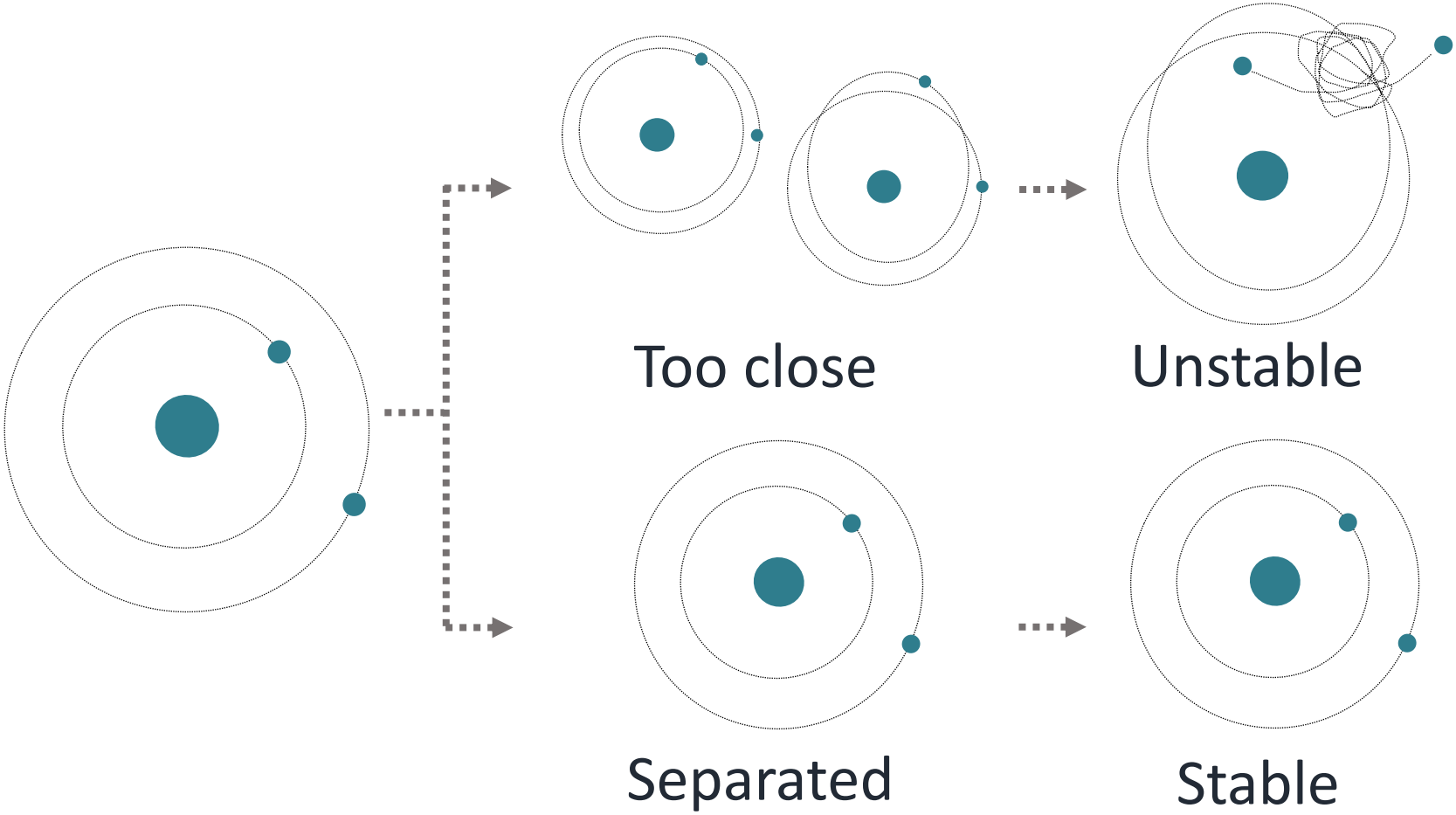




How to estimate the stability of these relativistic multi-body systems? How GR affect the stability?

Introduction Hill Stability Problem



Hill stability problem

: what is the initial condition that two orbit will not cross?

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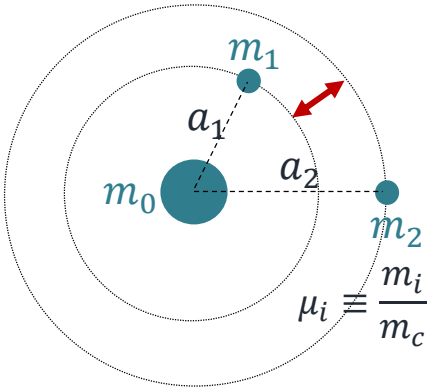
Numerical Simulation Method

Initial condition



Chambers et al. 1996

equal masses, circular orbits

$m_0 = 10^6 \mathrm{M}_\odot \quad \mu = 10^{-6} \quad e_1 = 0、\ e_2 = 0 \quad a_1 = 1.0$ AU

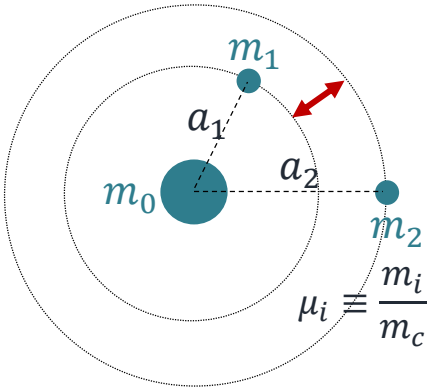
$$a_2 - a_1 = \Delta R'_{\mathrm{H}} \qquad R'_{\mathrm{H}} \equiv \left(\frac{\mu_1 + \mu_2}{3}\right)^{\frac{1}{3}} \frac{a_1 + a_2}{2}$$

Mutual Hill radius

Make many models by changing Δ

Numerical Simulation Method

Initial condition



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Mutual Hill radius

Make many models by changing Δ

Orbital evolution

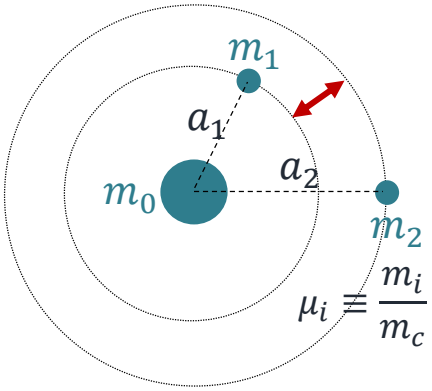
Calculate orbital evolution in both **Newtonian** and **First order post-Newtonian** equation of motion

$$\begin{aligned}
\frac{d\boldsymbol{v}_k}{dt} = &-G\sum_{n\neq k} m_n \frac{\boldsymbol{x}_k - \boldsymbol{x}_n}{|\boldsymbol{x}_k - \boldsymbol{x}_n|^3}\left[1 - 4\frac{G}{c^2}\sum_{n'\neq k}\frac{m_{n'}}{|\boldsymbol{x}_k - \boldsymbol{x}_{n'}|}\right. \\
&- \frac{G}{c^2}\sum_{n'\neq n}\frac{m_{n'}}{|\boldsymbol{x}_n - \boldsymbol{x}_{n'}|}\left\{1 - \frac{(\boldsymbol{x}_k - \boldsymbol{x}_n)\cdot(\boldsymbol{x}_n - \boldsymbol{x}_{n'})}{2|\boldsymbol{x}_n - \boldsymbol{x}_{n'}|^2}\right\} \\
&\left. + \left(\frac{|\boldsymbol{v}_k|}{c}\right)^2 + 2\left(\frac{|\boldsymbol{v}_n|}{c}\right)^2 - 4\frac{\boldsymbol{v}_k\cdot\boldsymbol{v}_n}{c^2} - \frac{3}{2}\left\{\frac{(\boldsymbol{x}_k - \boldsymbol{x}_n)}{|\boldsymbol{x}_k - \boldsymbol{x}_n|}\cdot\frac{\boldsymbol{v}_n}{c}\right\}^2\right] \\
&- \frac{G}{c^2}\sum_{n\neq k}\frac{m_n(\boldsymbol{v}_k - \boldsymbol{v}_n)}{|\boldsymbol{x}_k - \boldsymbol{x}_n|^3}(\boldsymbol{x}_k - \boldsymbol{x}_n)\cdot(3\boldsymbol{v}_n - 4\boldsymbol{v}_k) \\
&- \frac{7}{2}\frac{G^2}{c^2}\sum_{n\neq k}\frac{m_n}{|\boldsymbol{x}_k - \boldsymbol{x}_n|}\sum_{n'\neq n}\frac{m_{n'}(\boldsymbol{x}_n - \boldsymbol{x}_{n'})}{|\boldsymbol{x}_n - \boldsymbol{x}_{n'}|^3}
\end{aligned}$$

Numerical Simulation Method

Initial condition

Chambers et al. 1996



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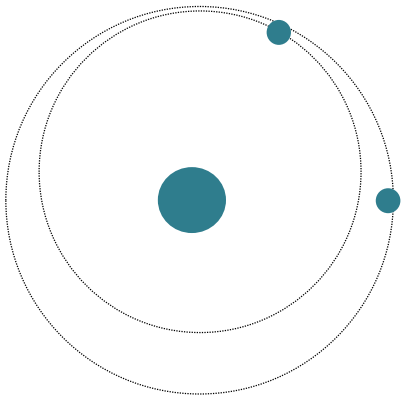
Mutual Hill radius

Make many models by changing Δ

Orbital evolution

Calculate orbital evolution in both **Newtonian** and **First order post-Newtonian** equation of motion

Orbits become too close



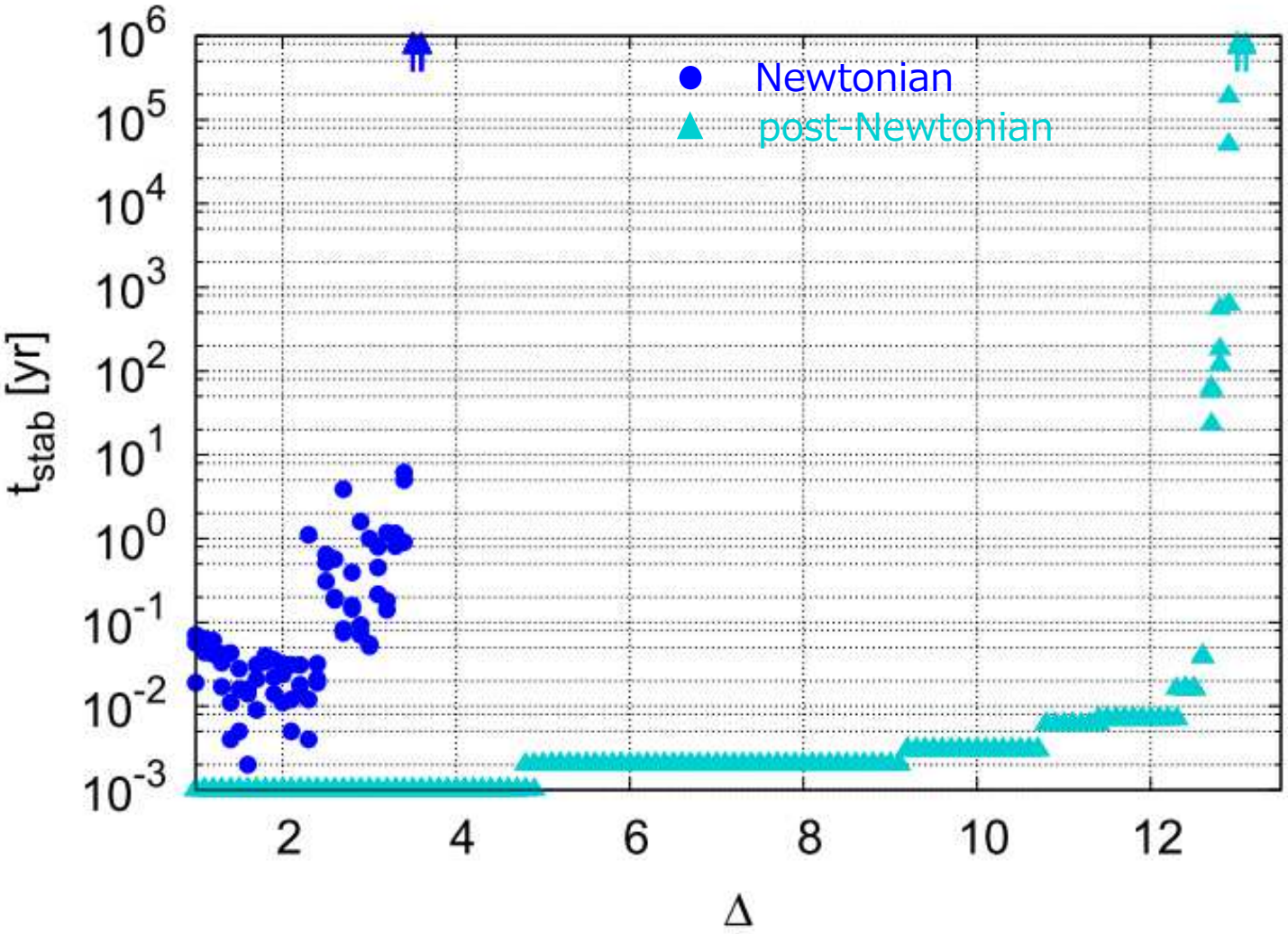
The system is unstable in the sense of Hill stability

$$\Leftrightarrow a_2(e_2 - 1) - a_1(e + 1) < R_H \qquad R_{\mathrm{H}} \equiv \left(\frac{\mu}{3}\right)^{\frac{1}{3}} a$$

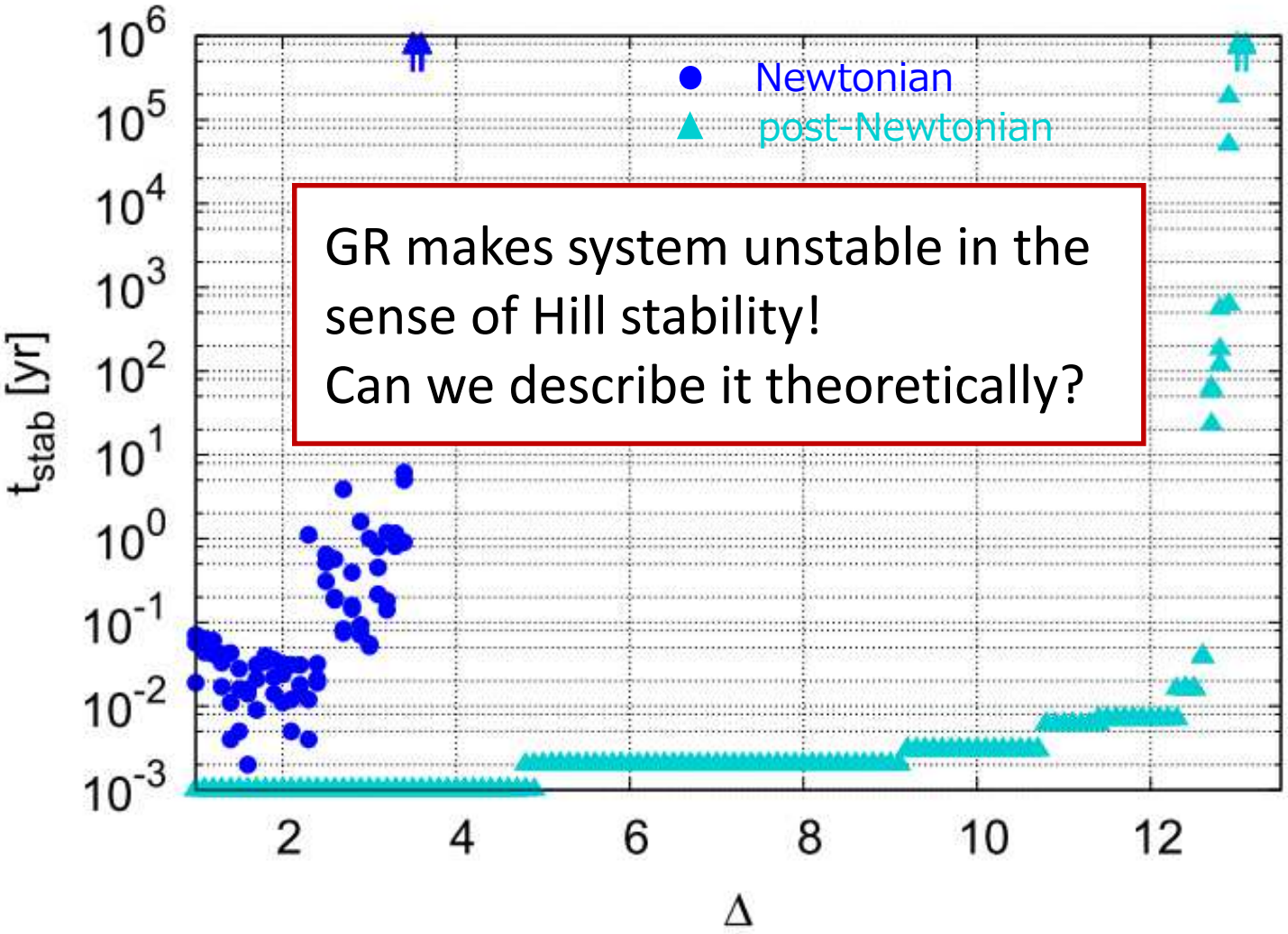
Hill radius

Detect time when above condition is satisfied as T_{stab}

Numerical Simulation Result



Numerical Simulation Result



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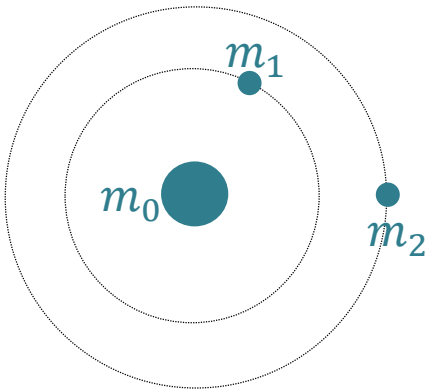
Theoretical Analysis Newtonian

Newtonian analysis of 3-body Hill stability (Marchal & Bozis (1982))

Sundman's inequality

$$\left(\sum_{i=0}^{2} m_i r_i^2\right)\left(\sum_{i=0}^{2} m_i v_i^2\right) \geq \left|\sum_{i=0}^{2} m_i \boldsymbol{r}_i \times \boldsymbol{v}_i\right|^2 + \left|\sum_{i=0}^{2} m_i \boldsymbol{r}_i \cdot \boldsymbol{v}_i\right|^2$$

$$\Rightarrow \boxed{\left(\sum_{i=0}^{2} m_i r_i^2\right) \cdot 2\big(\mathcal{H}_N - U(r)\big) \geq J^2}$$

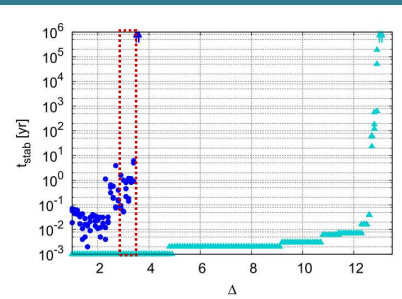


$\mathcal{H}_N = \frac{1}{2}\sum_{i=0}^{2} m_i v_i^2 + U(r)$: Newtonian Hamiltonian

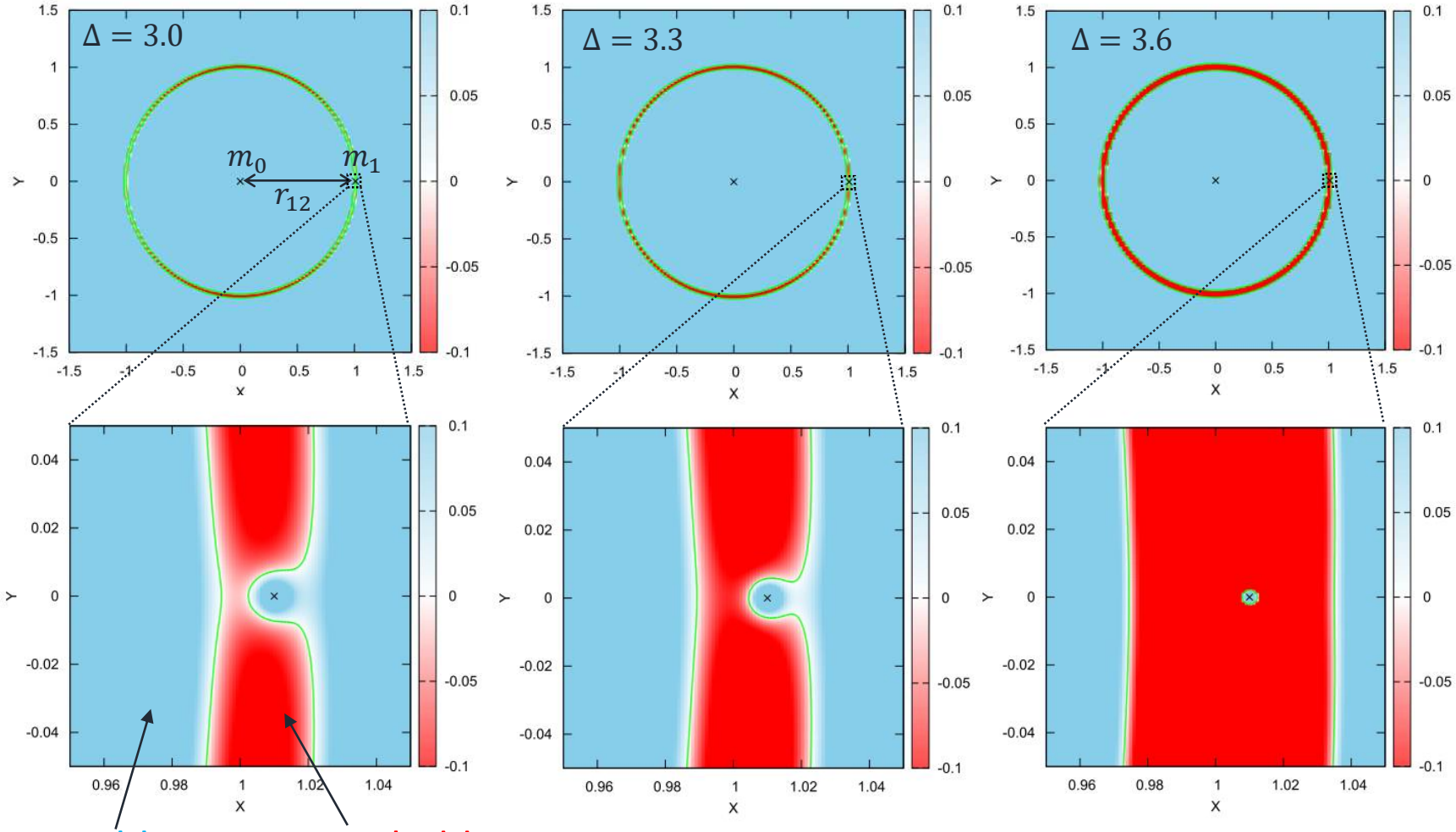
$J = \left|\sum_{i=0}^{2} m_i \boldsymbol{r}_i \times \boldsymbol{v}_i\right|$: total angular momentum

When the positions of m_0 and m_1 are fixed, this inequality gives accessible region of m_2

Theoretical Analysis Newtonian

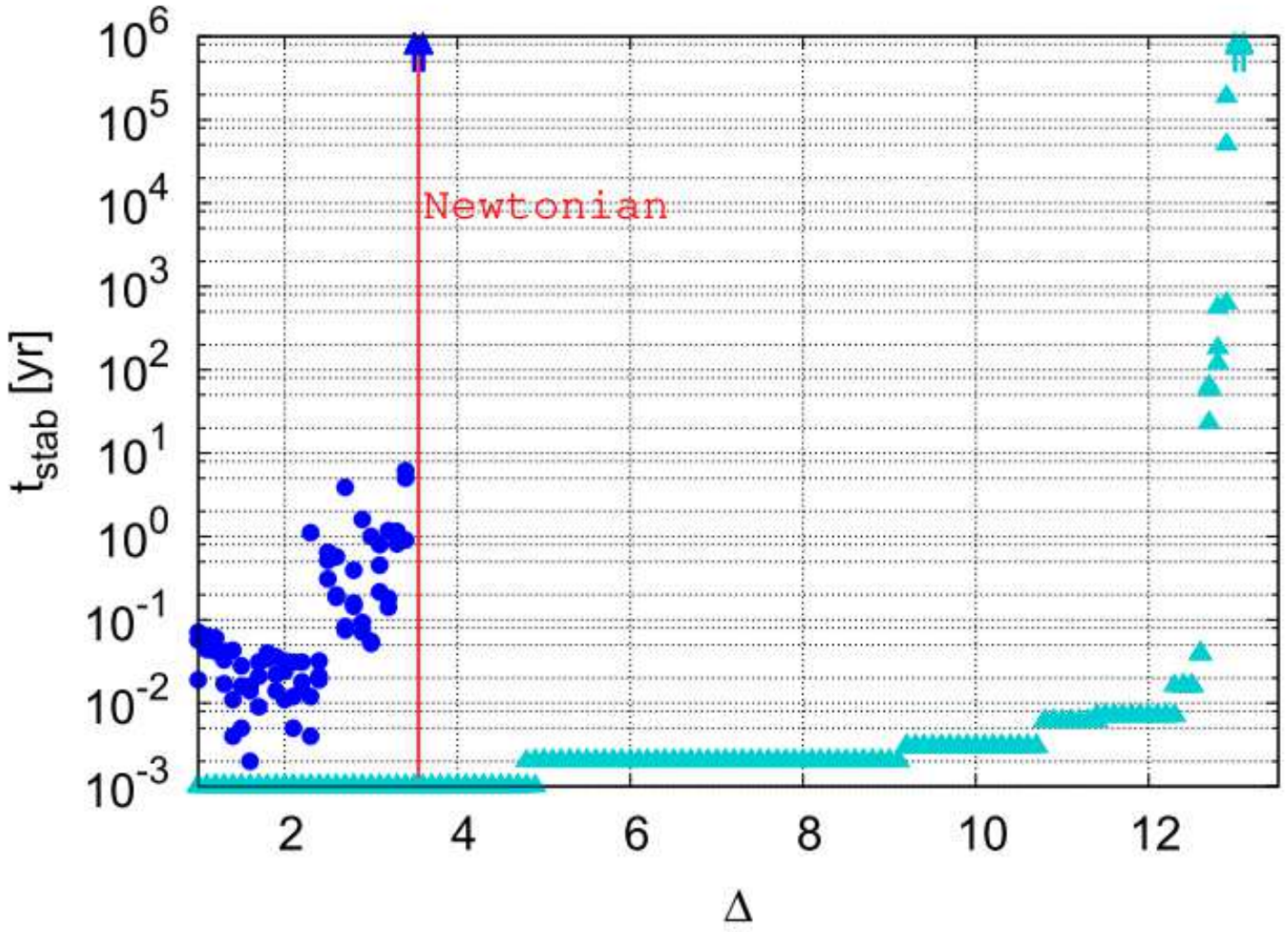


$r_{12} = 1.01a_1$

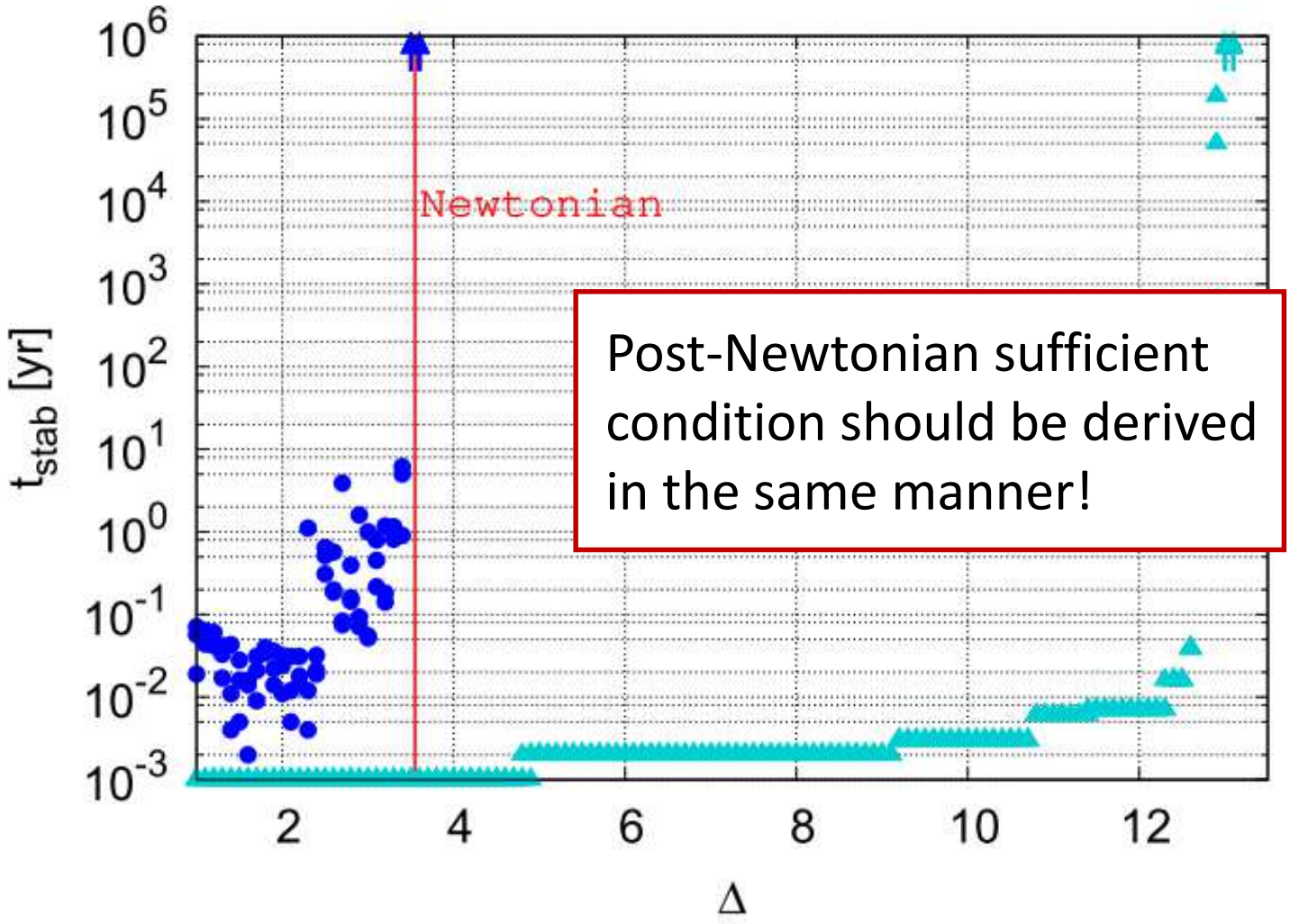


Accessible region Forbidden region

Theoretical Analysis Newtonian



Theoretical Analysis Newtonian



Post-Newtonian sufficient condition should be derived in the same manner!

Theoretical Analysis post-Newtonian

Post-Newtonian analysis

$$\mathcal{H}_{1\mathrm{PN}} \approx m_1 \mathcal{H}_{\mathrm{rel}}^{(0\to1)} + m_2 \mathcal{H}_{\mathrm{rel}}^{(0\to2)} - \frac{G m_1 m_2}{r_{12}}$$

$m_0 \gg m_1, m_2$
$\Rightarrow$ m_0 does not move from coordinate origin

$\mathcal{H}_{\mathrm{rel}}^{(0\to i)}$: 1st order post-Newtonian Hamiltonian of relative motion

Richardson & kelly (1988)

$$\mathcal{H}_{\mathrm{rel}}^{(0\to i)} = \frac{1}{2}\boldsymbol{p}_{0i}\cdot\boldsymbol{p}_{0i} - \frac{G m_0}{r_{0i}} - \frac{1}{c^2}\left[\sigma_0(\boldsymbol{p}_{0i}\cdot\boldsymbol{p}_{0i})^2 + \frac{\sigma_1}{r_{0i}}\boldsymbol{p}_{0i}\cdot\boldsymbol{p}_{0i} + \frac{\sigma_2}{r_{0i}^2} + \frac{\sigma_3}{r_{0i}^3}(\boldsymbol{r}_{0i}\cdot\boldsymbol{p}_{0i})^2\right]$$

$$\boldsymbol{p}_{0i} = \boldsymbol{v}_{0i} + \frac{1}{c^2}\left[4\sigma_0 v_{0i}^2 \boldsymbol{v}_{0i} + \frac{2\sigma_1}{r_{0i}}\boldsymbol{v}_{0i} + \frac{2\sigma_3}{r_{0i}^3}(\boldsymbol{r}_{0i}\cdot\boldsymbol{p}_{0i})\boldsymbol{r}_{0i}\right]$$: specific momentum

$$J = \left|\sum_{i=0}^{2} m_i \boldsymbol{r}_{0i}\times\boldsymbol{p}_{0i}\right|$$: total angular momentum

$$\sigma_0 = \frac{1-3\mu_i}{8} \qquad \sigma_1 = \frac{G m_0(3+\mu_i)}{2}$$

$$\sigma_2 = -\frac{G^2 m_0^2}{2} \qquad \sigma_3 = \frac{G m_0 \mu_i}{2}$$

Theoretical Analysis post-Newtonian

Post-Newtonian analysis

Sundman's inequality

$$\left(\sum_{i=0}^{2} m_i r_{0i}^2\right)\left(\sum_{i=0}^{2} m_i p_{0i}^2\right) \geq \left|\sum_{i=0}^{2} m_i \boldsymbol{r}_{0i} \times \boldsymbol{p}_{0i}\right|^2 + \left|\sum_{i=0}^{2} m_i \boldsymbol{r}_{0i} \cdot \boldsymbol{p}_{0i}\right|^2$$

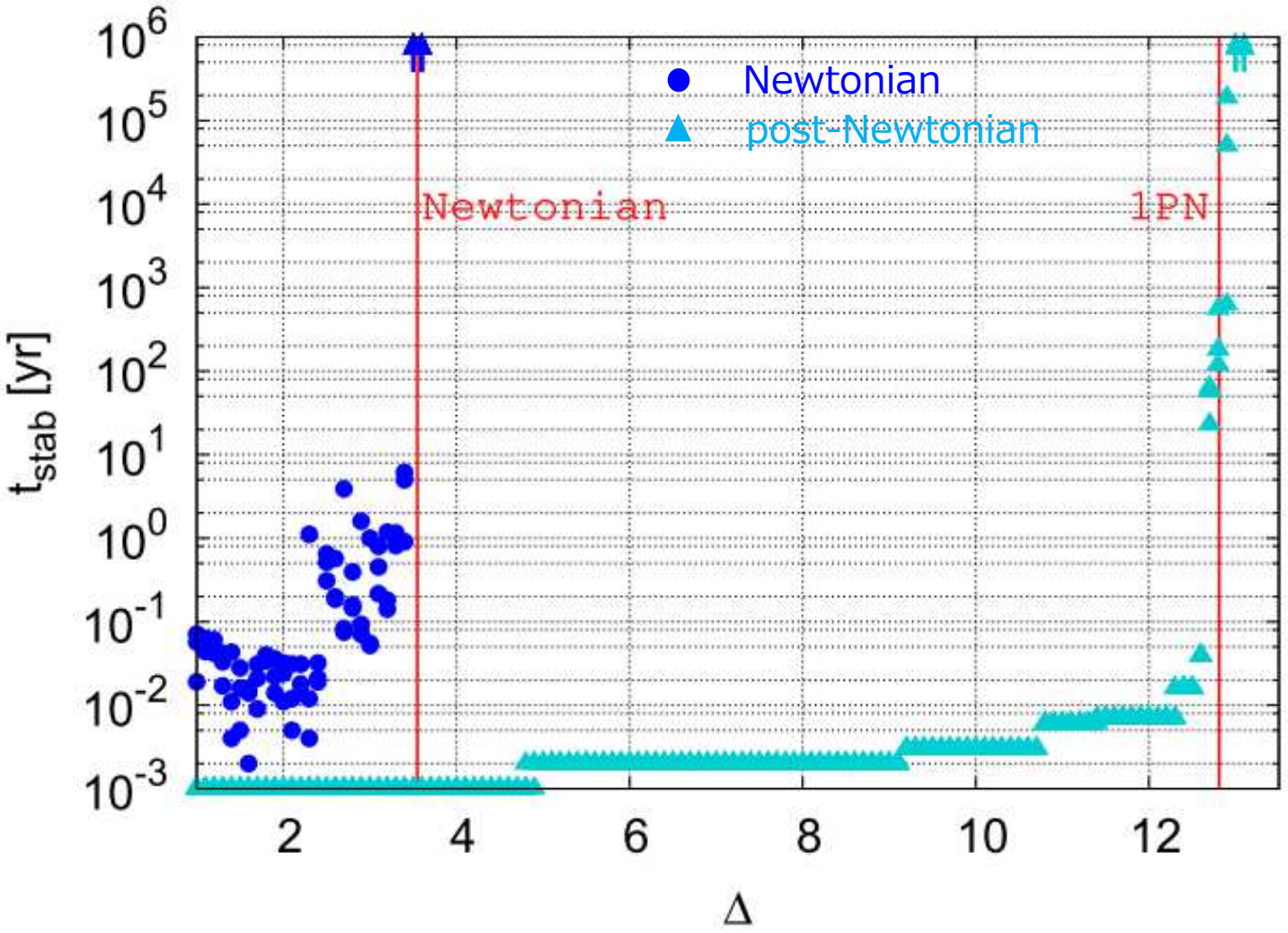
$$\Rightarrow \left(\sum_{i=0}^{2} m_i r_{0i}^2\right) \cdot 2\left(\mathcal{H}_{1\mathrm{PN}} - U(r) + \frac{1}{c^2}\sum_{i=1}^{2}\left[\sigma_0(\boldsymbol{p}_{0i}\cdot\boldsymbol{p}_{0i})^2 + \frac{\sigma_1}{r_{0i}}\boldsymbol{p}_{0i}\cdot\boldsymbol{p}_{0i} + \frac{\sigma_2}{r_{0i}^2} + \frac{\sigma_3}{r_{0i}^3}(\boldsymbol{r}_{0i}\cdot\boldsymbol{p}_{0i})^2\right]\right) \geq J^2$$

$v_{0i}^2 \approx \dfrac{Gm_0}{r_{0i}}$: virial approximation

$$\left(\sum_{i=0}^{2} m_i r_{0i}^2\right) \cdot 2\left(\mathcal{H}_{1\mathrm{PN}} - U(r) + \frac{9G^2 m_0^2}{8c^2}\sum_{i=1}^{2}\frac{m_i^2}{r_{0i}^2}\right) \geq J^2$$

this inequality gives accessible region of m_2

Theoretical Analysis



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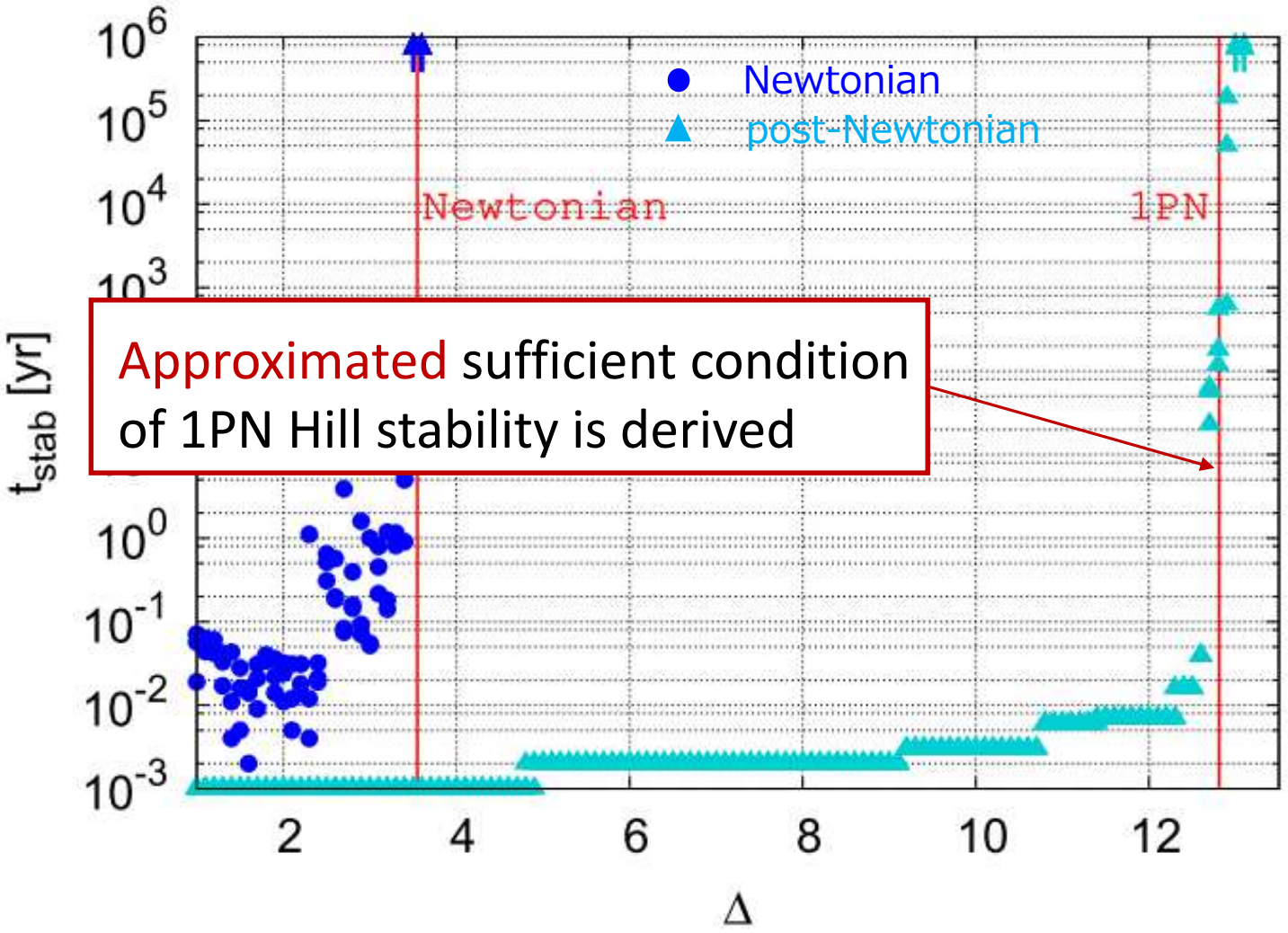

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Summary and Future Work

- GR makes 3-body system unstable in the sense of Hill stability
- Approximated sufficient condition of 1st order post-Newtonian Hill stability is described with virial approximation

- 4-body or more than 4-body system
  →Nekhoroshev‘s theorem, mean motion resonance
- Higher order post-Newtonian effect
- Inclined system→Kozai-Lidov mechanism