Rotational equilibria on the 2D Lagrange coordinates

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Collaborators:

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Yu Yamamoto (Waseda)

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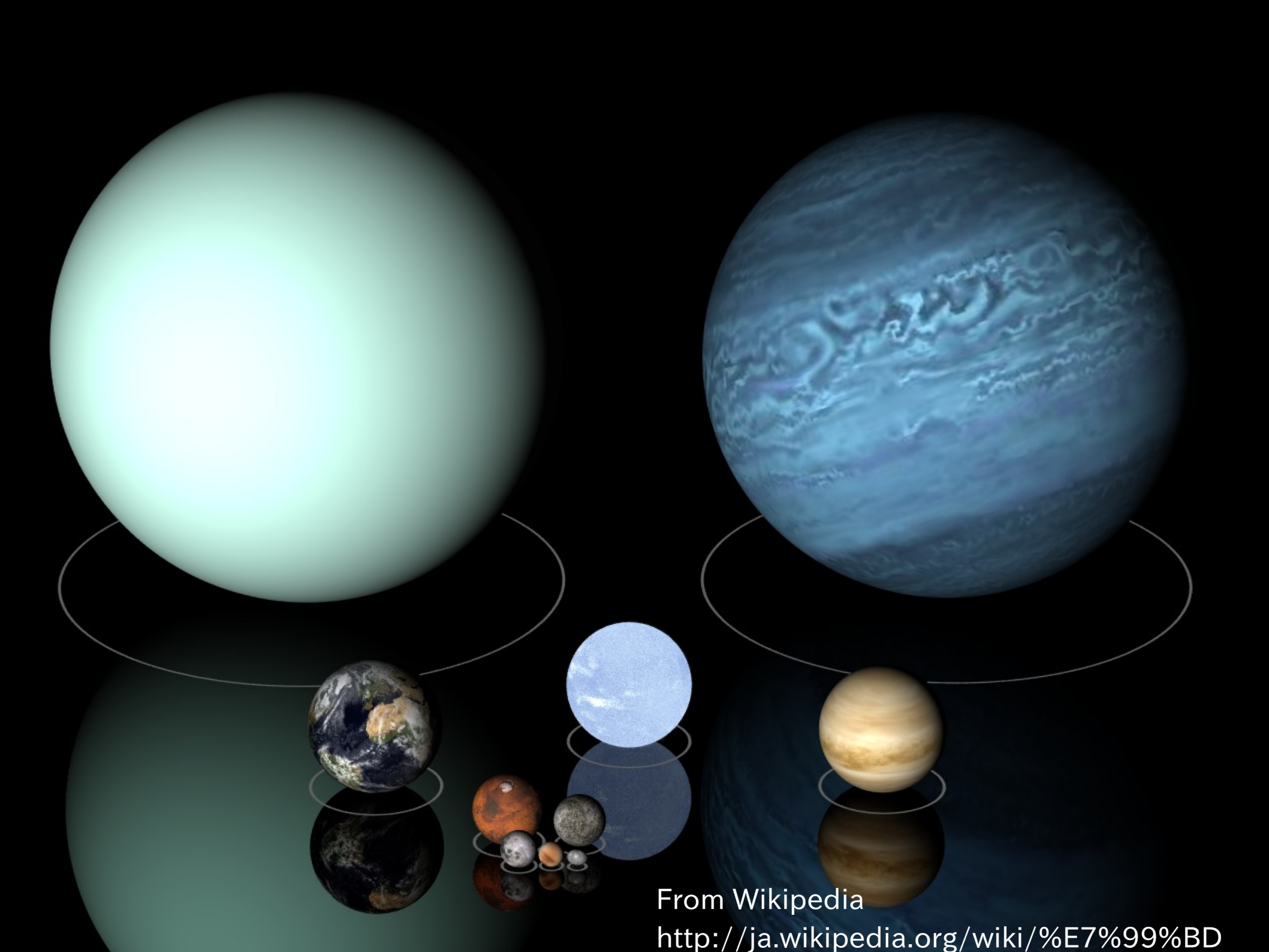
Outline

- Introduction
- Rotating stars
- Observations of rotating stars
- Rotational equilibria on the 2D Lagrange coordinate
- New numerical scheme “W4 method”
- Spectral method for 2D Lagrange coordinate



From Wikipedia
http://ja.wikipedia.org/wiki/%E7%99%BD

Force balance of star

$$\frac{1}{\rho}\nabla p = -\nabla \phi_g$$

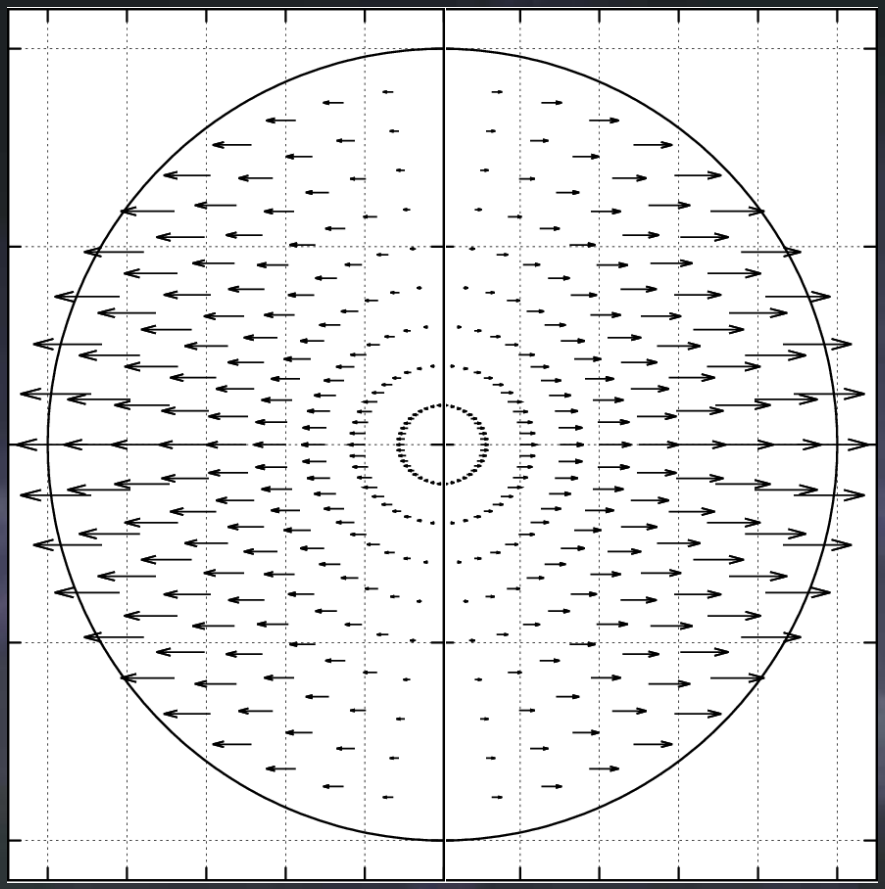
density

pressure

gravity

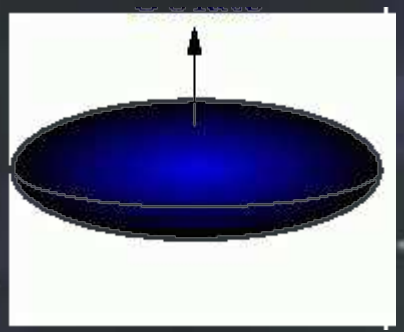
A star is roughly spherical...

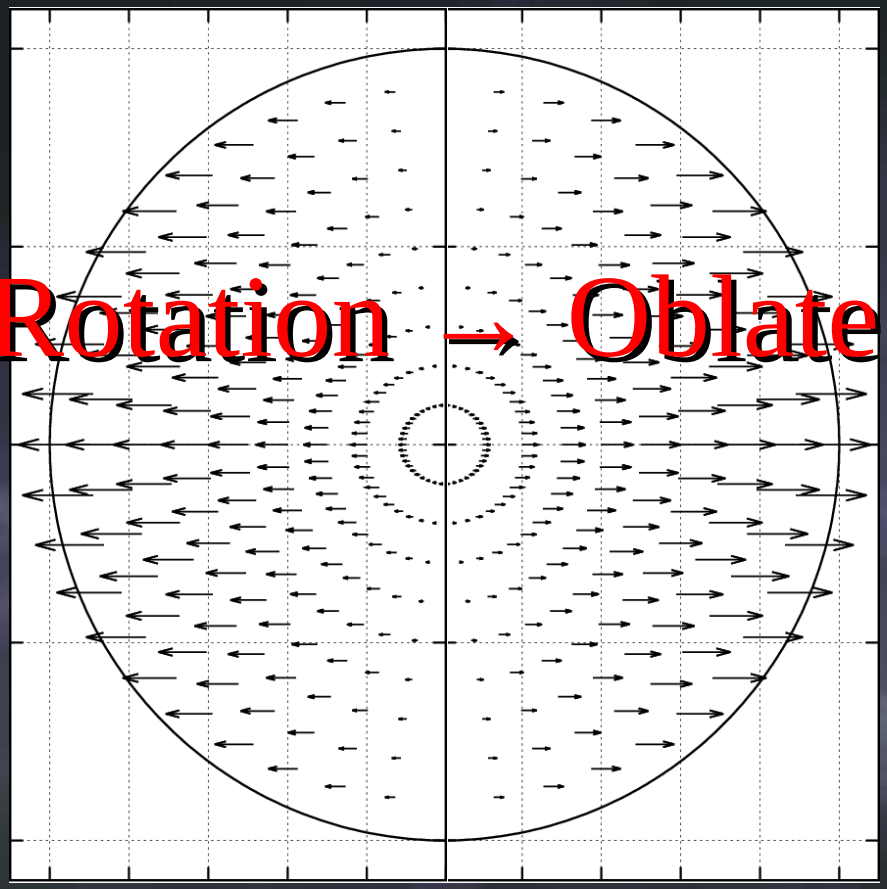
Centrifugal force



$$\frac{1}{\rho}\nabla p = -\nabla\phi_g + \Omega^2 R\boldsymbol{e}_R$$

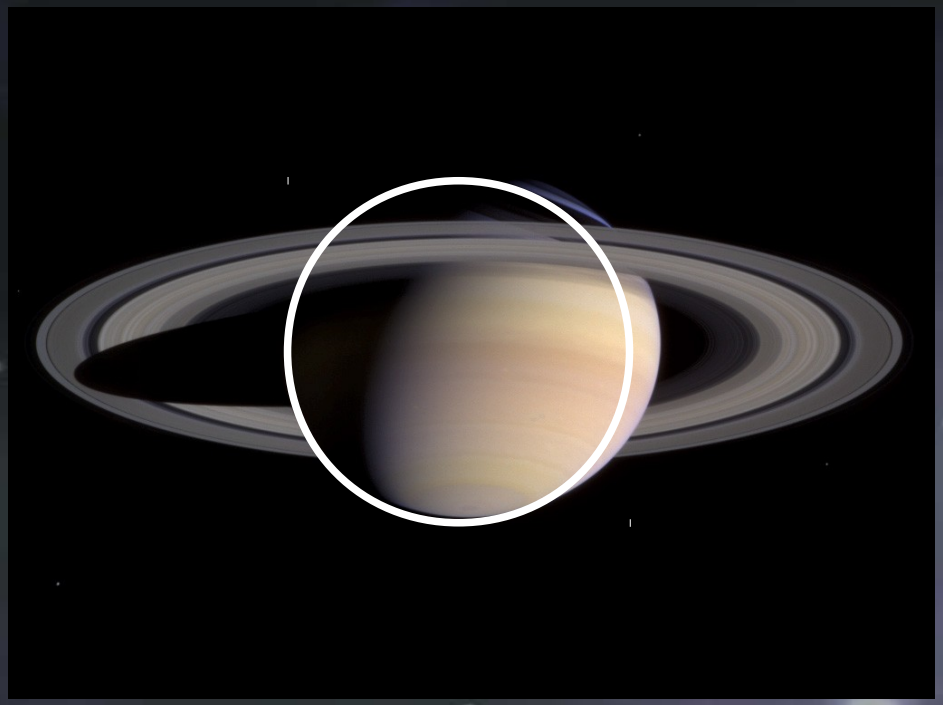
Centrifugal force

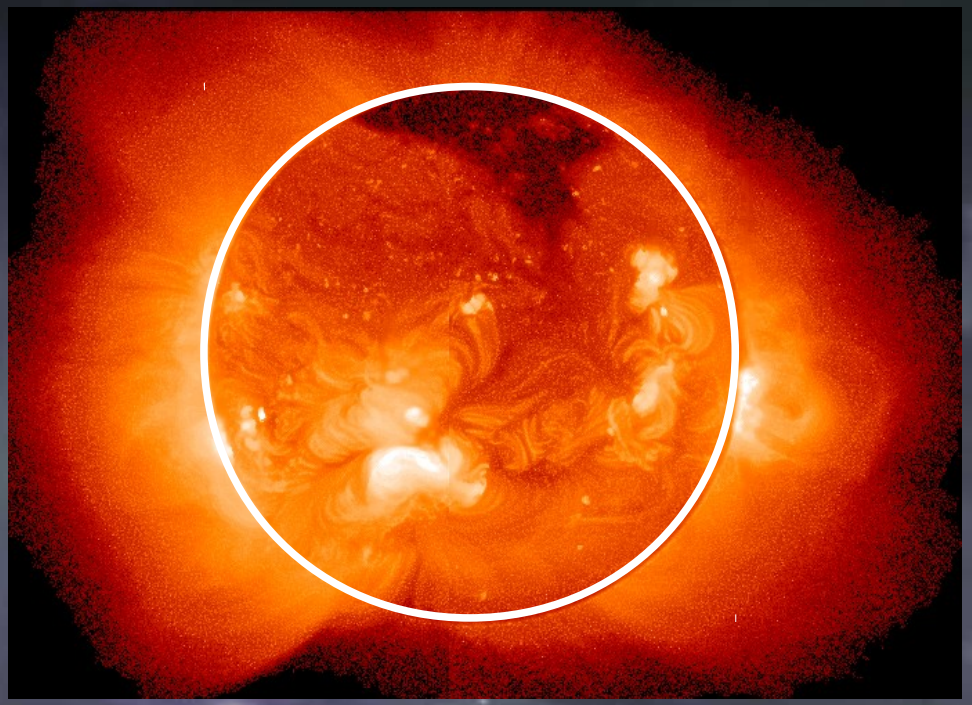




$$\frac{1}{\rho}\nabla p = -\nabla\phi_g + \Omega^2 R\boldsymbol{e}_R$$

Shape





- Saturn
- P ～ 10 hr

Sun

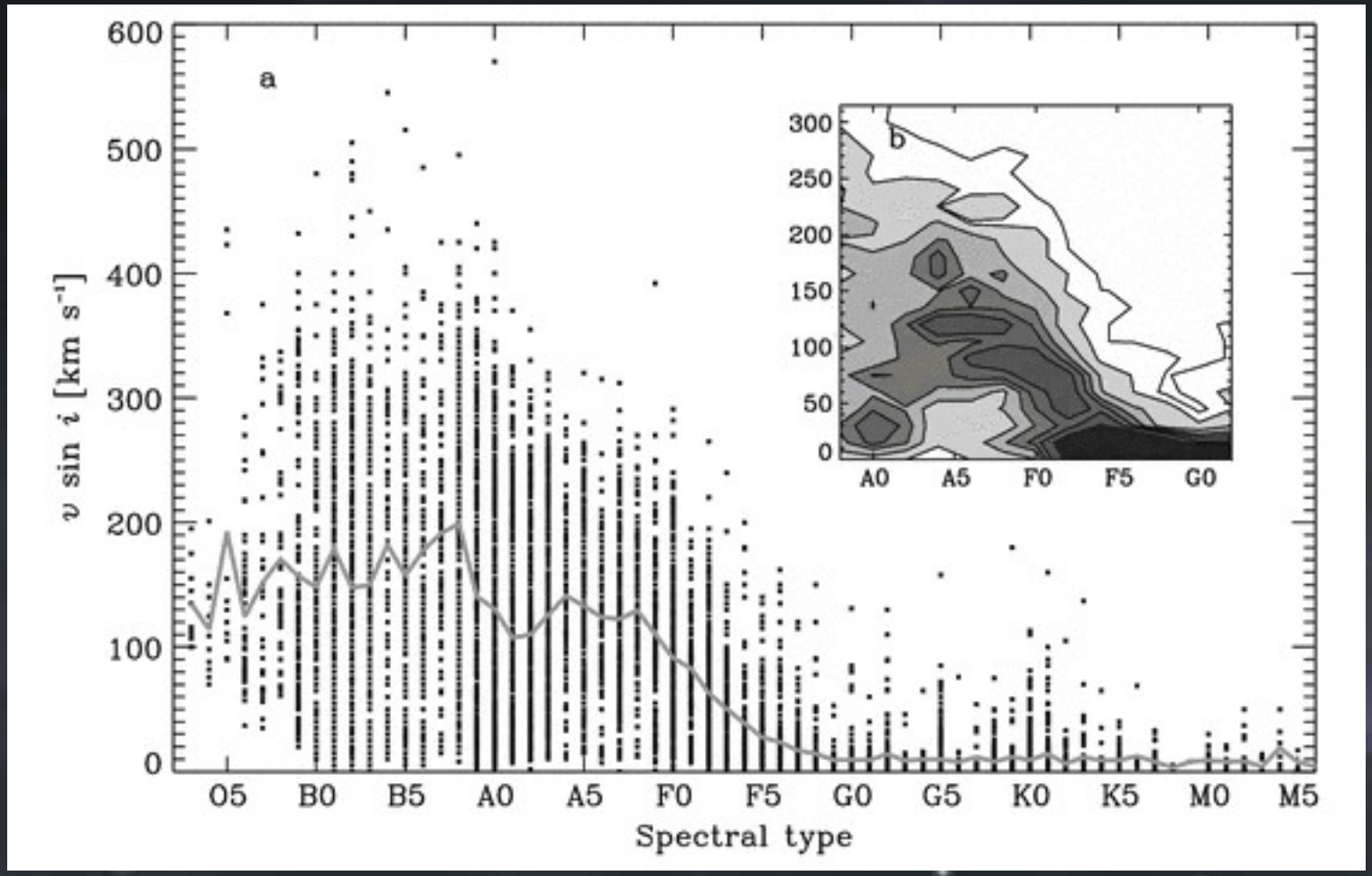
P ～ 28 days

oblateness ≡ $(R_{eq} - R_{pol}) / R_{eq}$

oblateness ～ 9.8×10^{-2}～ $(5.00 \pm 0.34)\times10^{-6}$

(Kuhn et al. 2012; Fujisawa & Eriguchi 2014)

Rotation velocities of stars



Royer (2009)

High mass stars (short life) -> Fast rotator

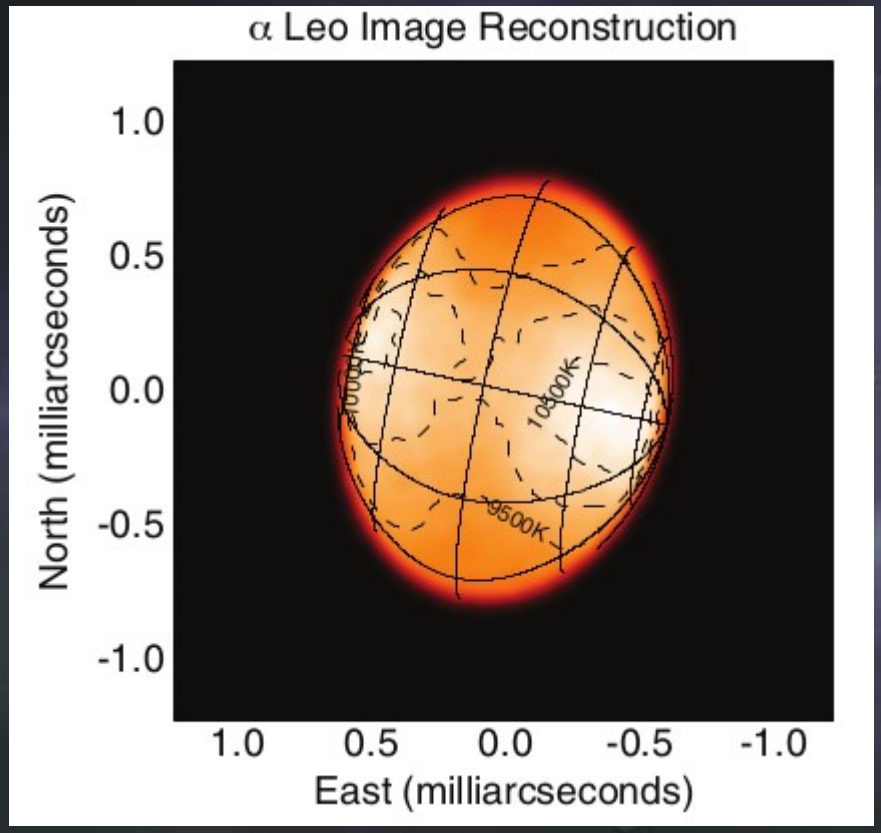
Low mass stars (long life)-> Slow rotator

Stellar shape of rotating star

Che et al.(2009)

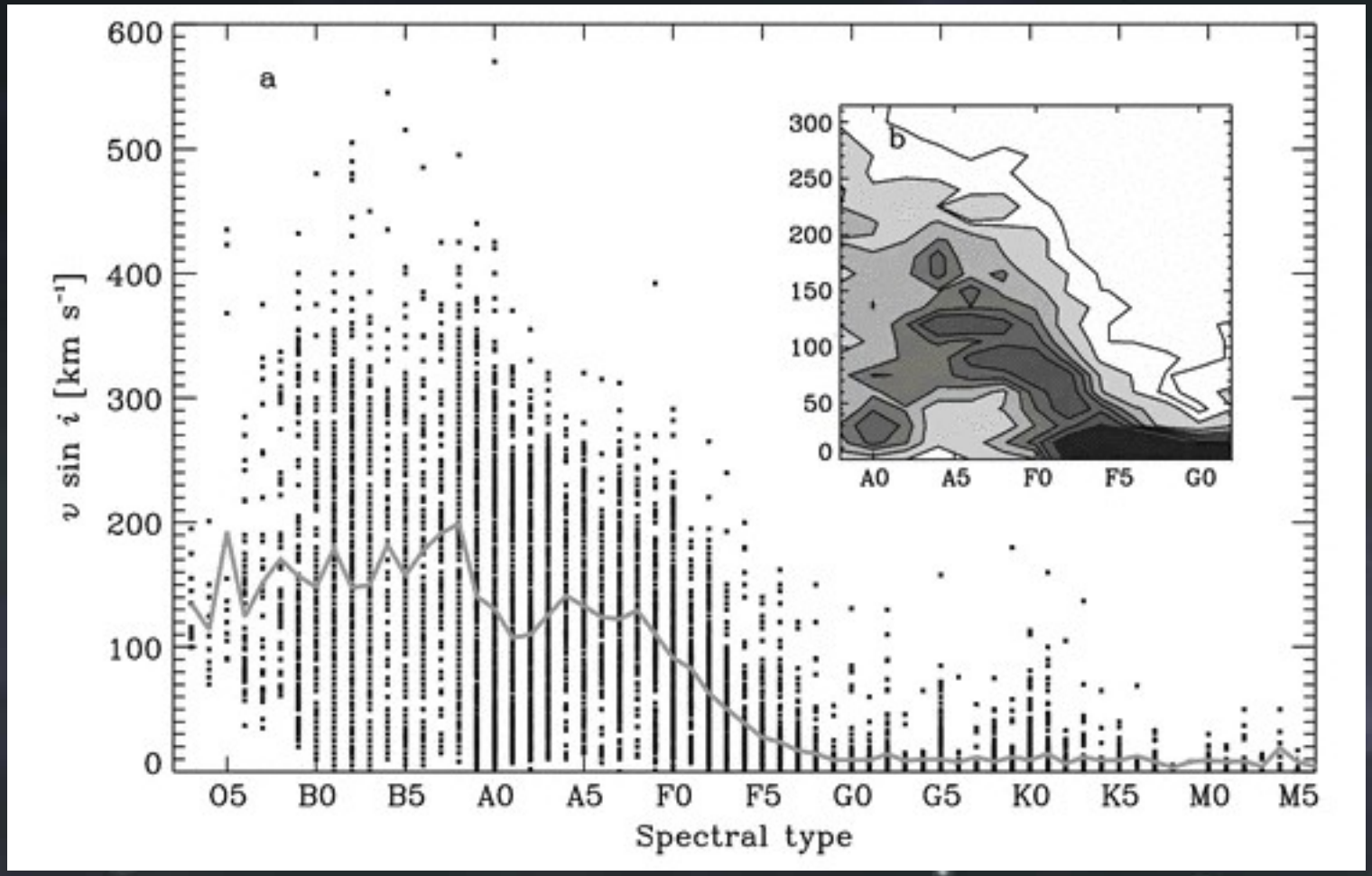
Oblate.～ 0.23, v sin i ～ 320 km/s

Regulus



- Image reconstruction from optical interferometers (Georgia State University CHARA interferometer array)

Rotation velocities of stars

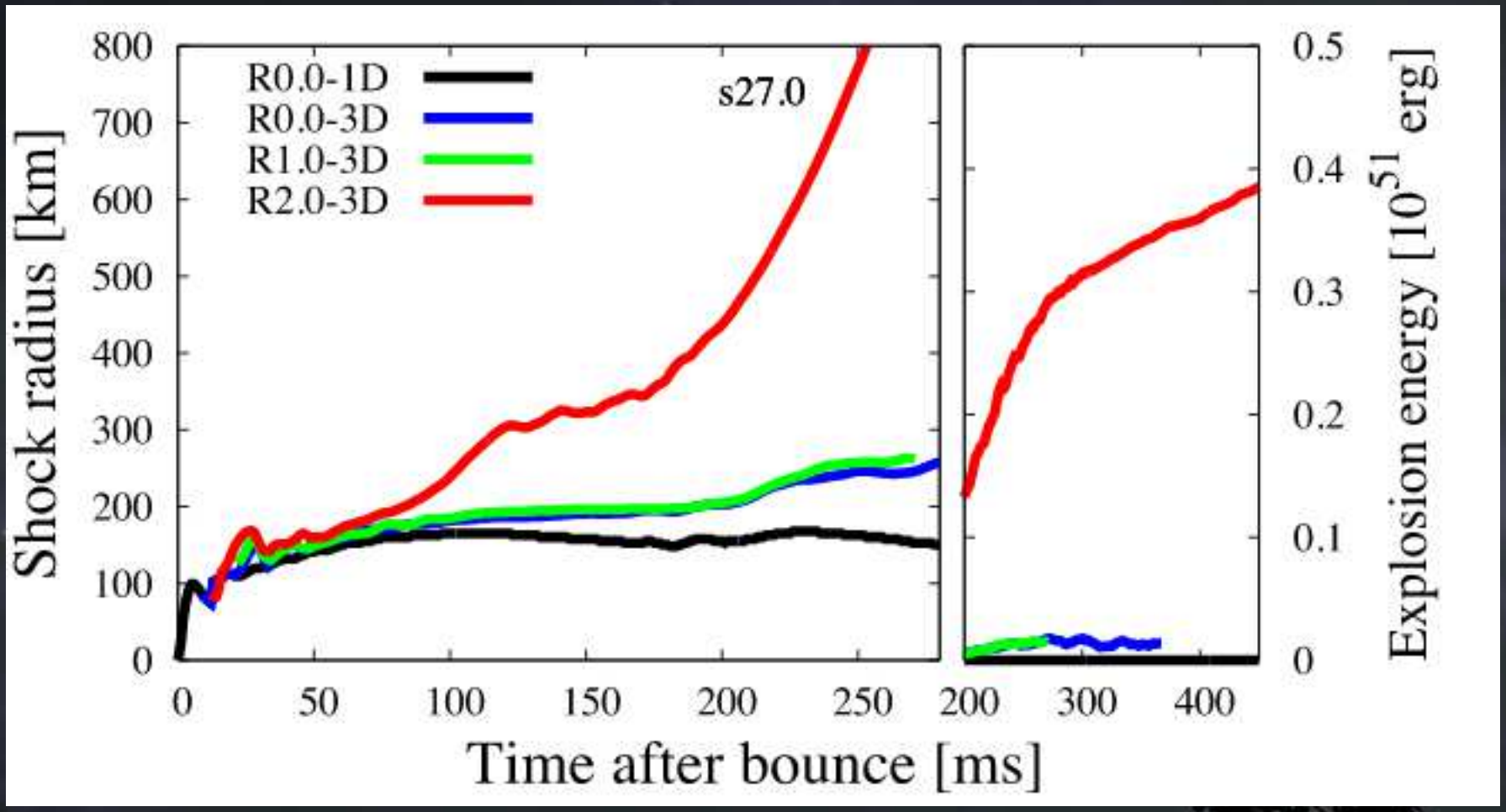


High mass stars (short life) -> Fast rotator

Low mass stars (long life) -> Slow rotator

Royer (2009)

Core-collapse supernovae



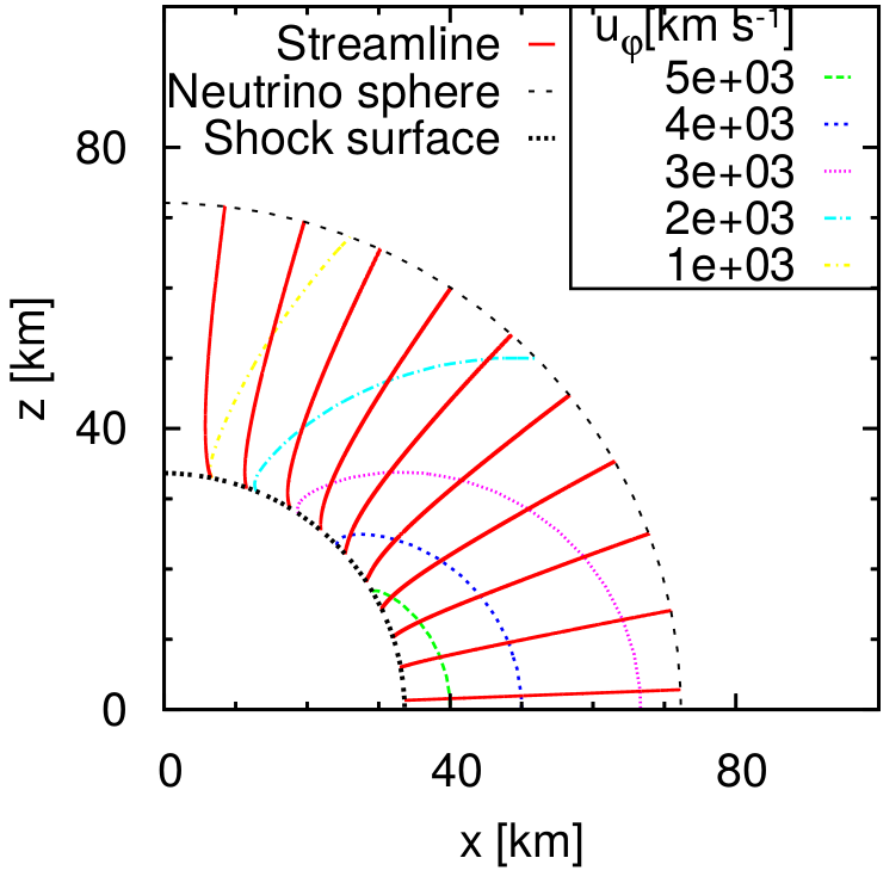
Takiwaki et al. (2016)

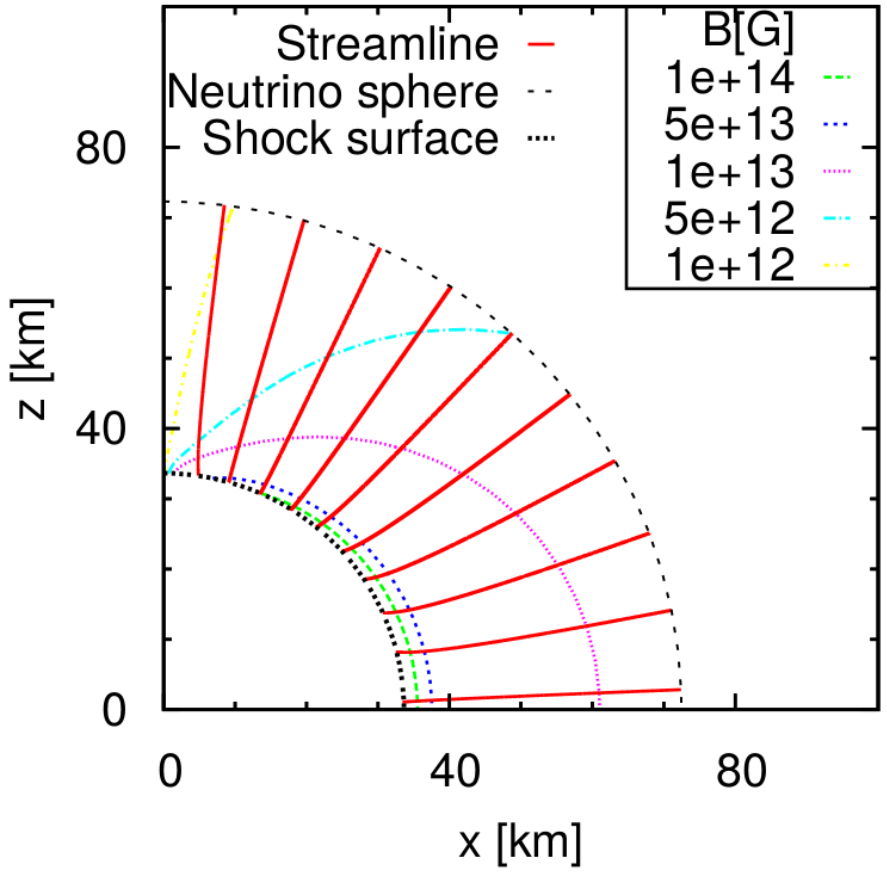
Stalled shock with rotation and magnetic field in 2D

Fujisawa et al. (2019)

With rotation

With toroidal field

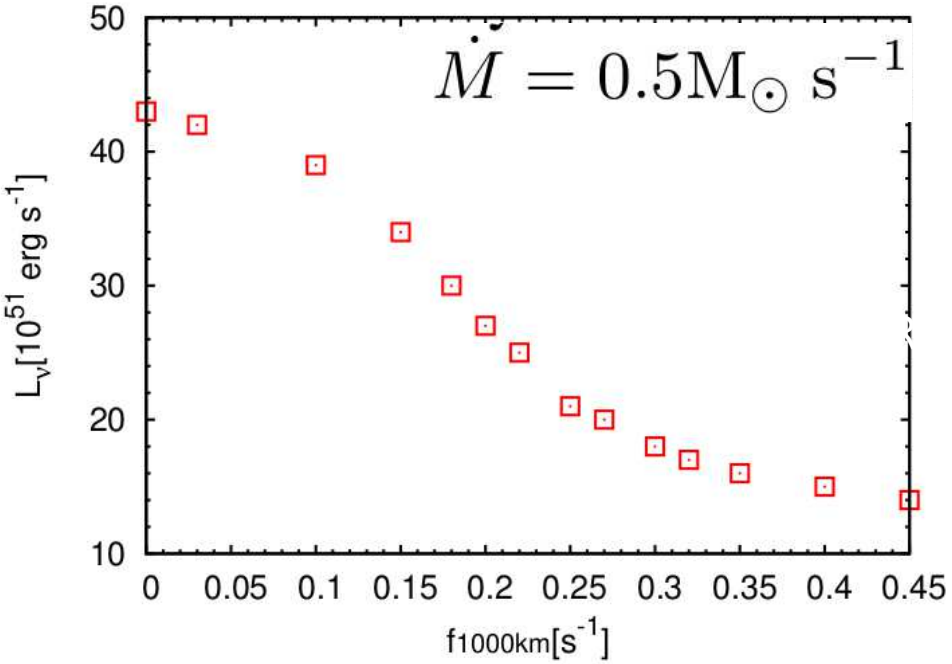




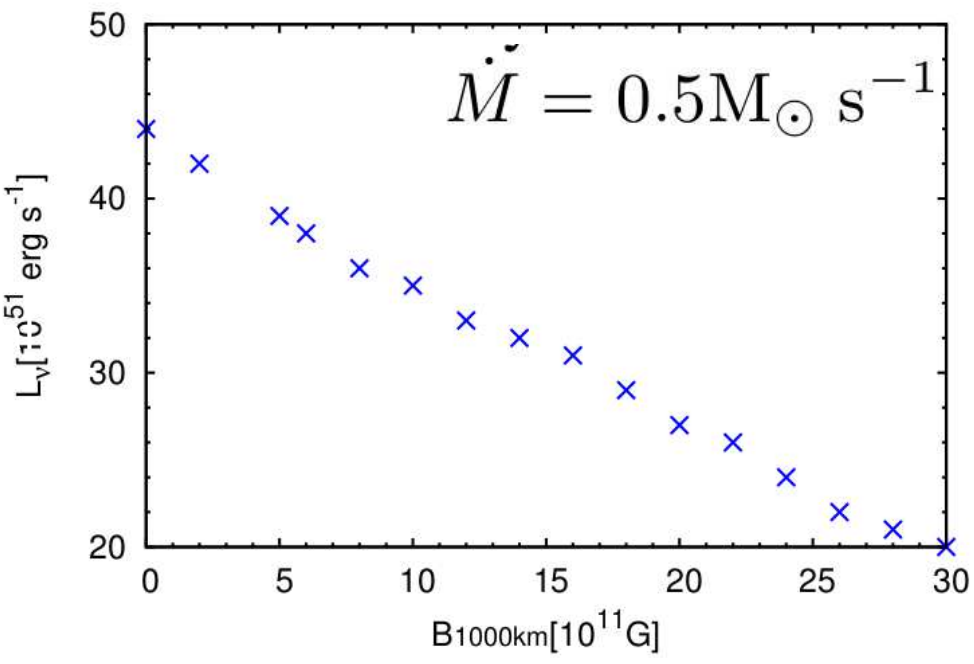
Critical Neutrino luminosity with rotation and magnetic field

Fujisawa et al. (2019)

Rotation



Toroidal field



- A critical neutrino luminosity, beyond which there exits no steady solution, and the shock might revival (Burrows & Goshy 1993).
- Reduce by 70% (rotation)/50% (toroidal field)

Stellar evolution models

- Non-rotating model

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \varrho} ,$$

$$\frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^4} ,$$

$$\frac{\partial l}{\partial m} = \varepsilon_{\mathrm{n}} - \varepsilon_{\nu} - c_P \frac{\partial T}{\partial t} + \frac{\delta}{\varrho} \frac{\partial P}{\partial t} ,$$

$$\frac{\partial T}{\partial m} = -\frac{GmT}{4\pi r^4 P} \nabla ,$$

Spherical averaged 1D rotating model

$$\frac{dP}{dM_P} = -\frac{GM_P}{4\pi r_P^4} f_P , \qquad \frac{dr_P}{dM_P} = \frac{1}{4\pi r_P^2 \overline{\varrho}} ,$$

$$\frac{dL_P}{dM_P} = \varepsilon_{\mathrm{nucl}} - \varepsilon_{\nu} + \varepsilon_{\mathrm{grav}} ,$$

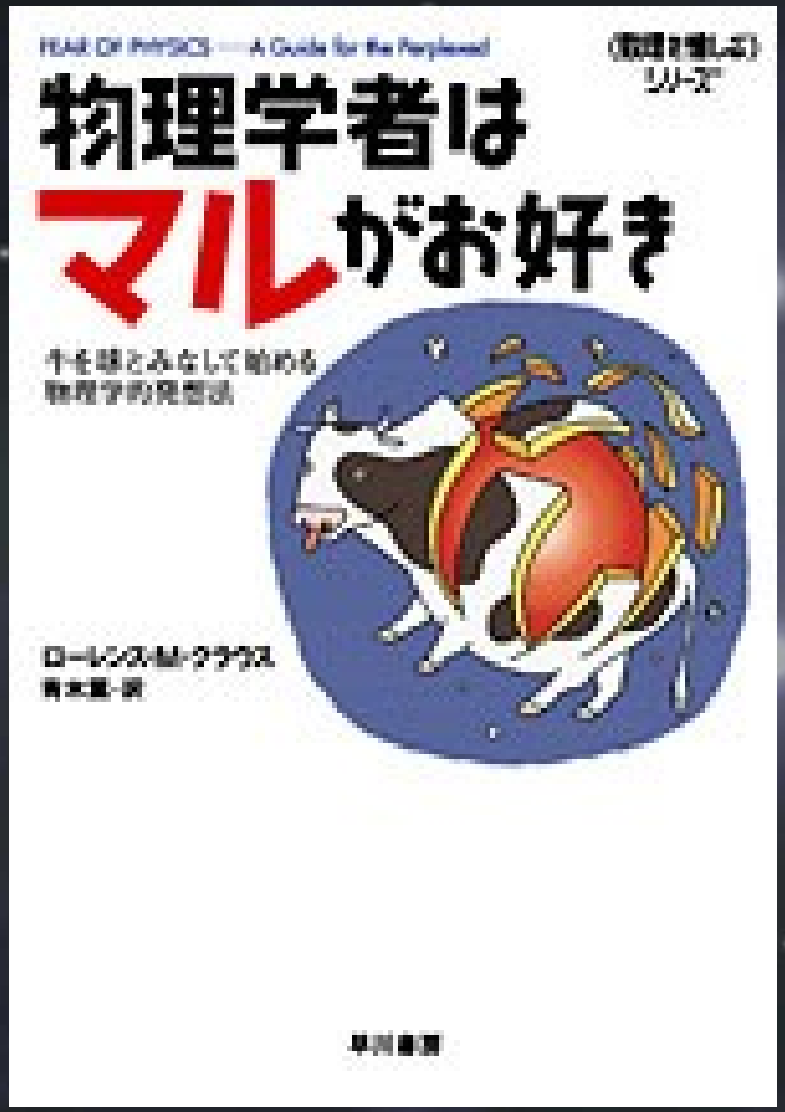
$$\frac{d\ln T}{dM_P} = -\frac{GM_P}{4\pi r_P^4} f_P \min\left[\nabla_{\mathrm{ad}}, \nabla_{\mathrm{rad}} \frac{f_T}{f_P}\right] ,$$

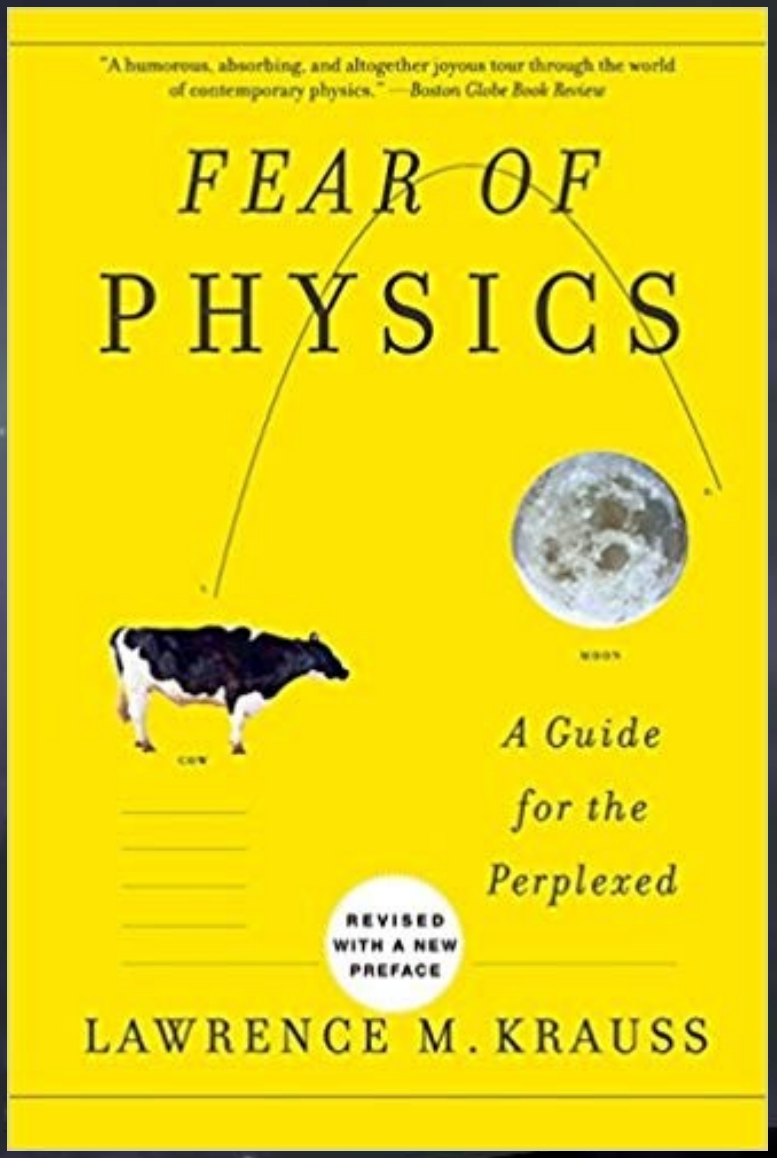
$$\text{where} \quad f_P = \frac{4\pi r_P^4}{GM_P S_P} \frac{1}{< g_{\mathrm{eff}}^{-1} >}$$

$$\text{and} \quad f_T = \left(\frac{4\pi r_P^2}{S_P}\right)^2 \frac{1}{< g_{\mathrm{eff}} >< g_{\mathrm{eff}}^{-1} >} .$$

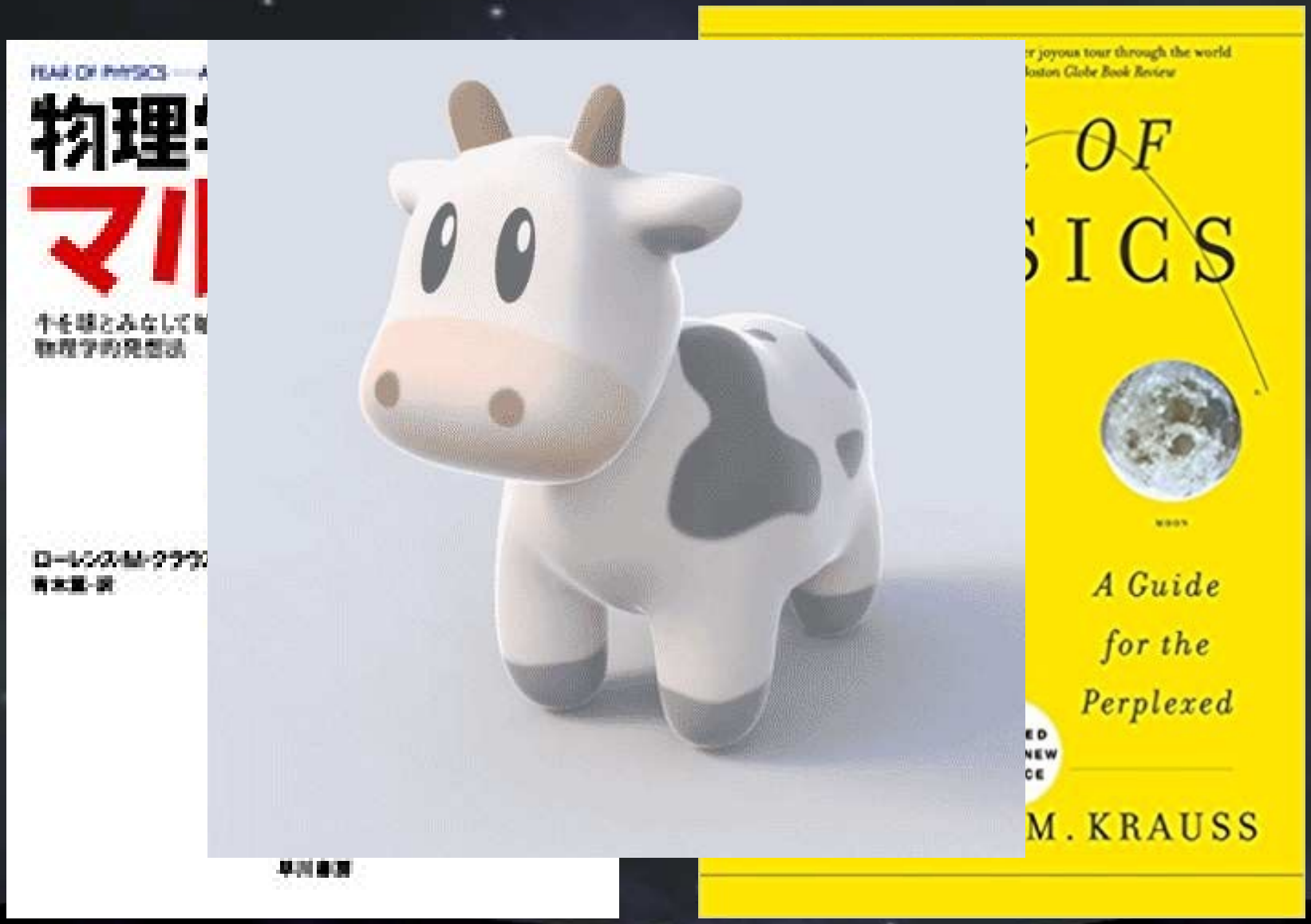
From “Physics, Formation and Evolution of Rotating Star”

Spherical cow





Spherical cow



- To construct fully 2D ~~cow~~ stellar evolution models with rapid rotation,,,
  - →2D formulation on the Lagrangian coordinate.
  - →Accurate and stable numerical scheme.

Previous works of 2D rapidly rotating stars

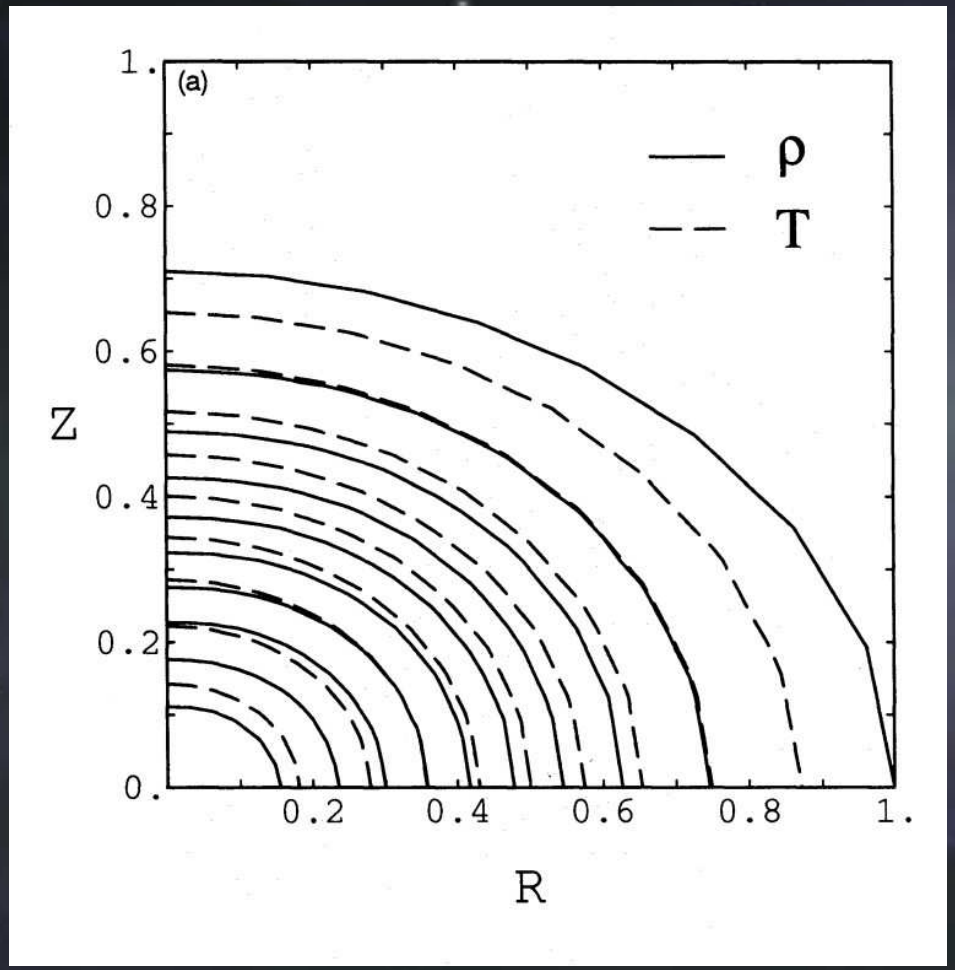
⇒2D Eulerian coordinates

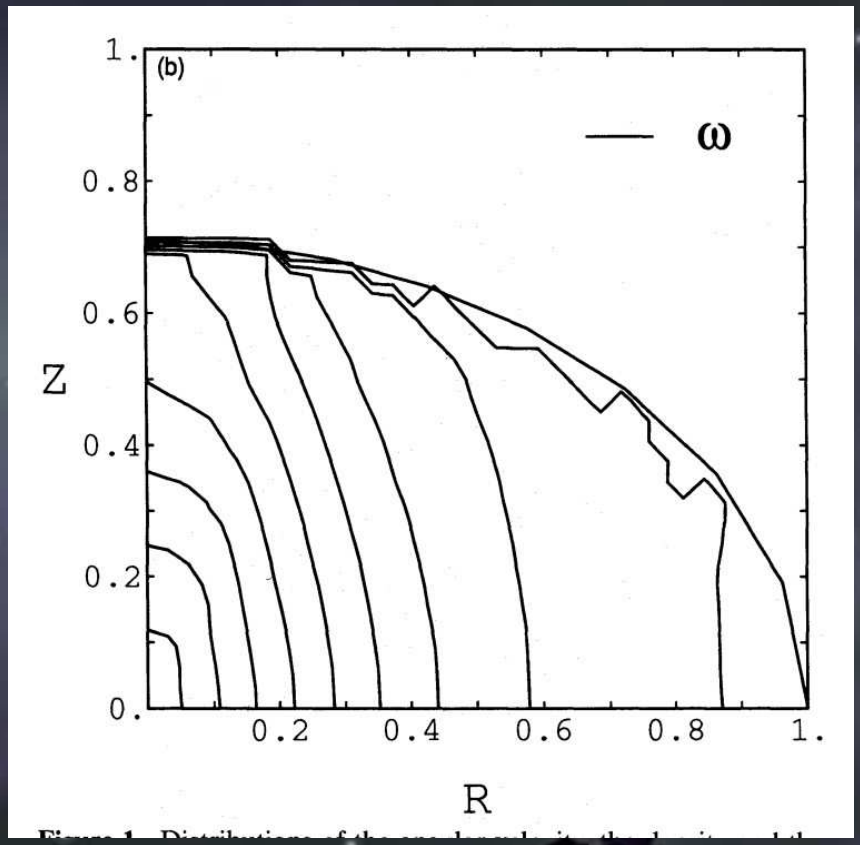
- Finite difference method (Uryu & Eriguchi 1994, Fujisawa 2015)
- Spectral method：ESTAR code
  (Espinosa & Rieutord 2007; 2013, Bouchaud et al. 2019)
- 2D evolution models by ESTAR code (Gagnier et al. 2019)

⇒2D Lagrangian coordinate

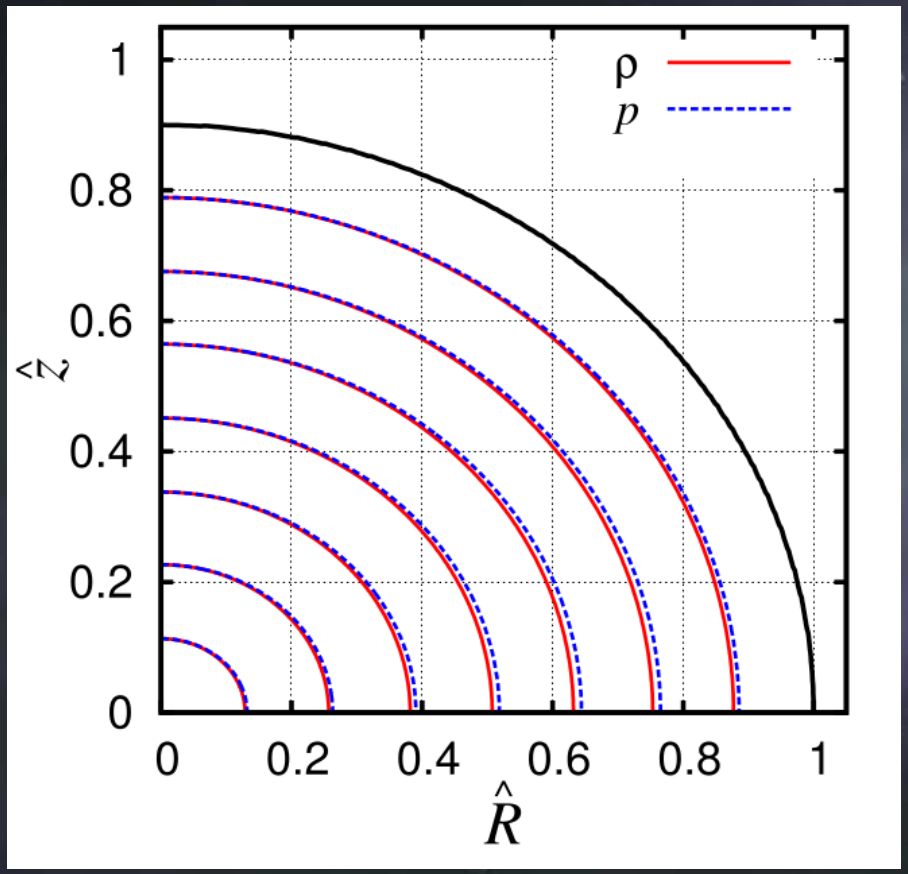
- Weak solutions using triangle meshes
  (Yasutake, Fujisawa, Yamada 2014,2016)
- Finite difference method (Ogata et al. Ogata's poster)
- 2D spectral method (This talk)

Uryu et al. (1994,1995)

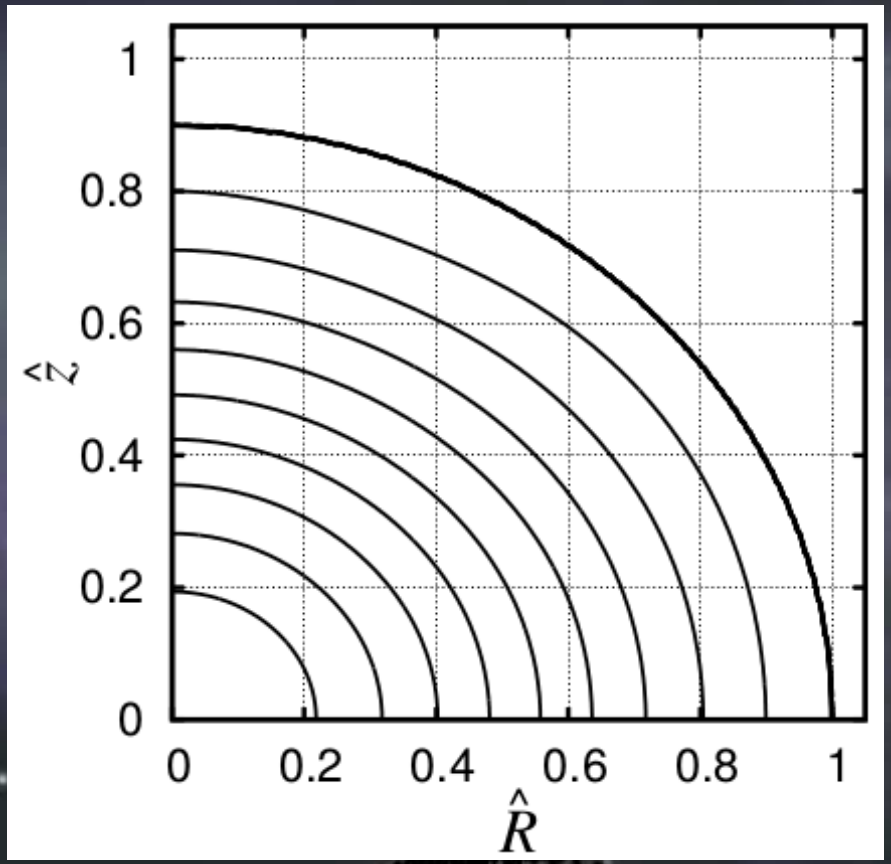




Shellular rotating star (Fujisawa 2015)



Ω



ESTER

Evolution STEllaire en Rotation

Project Description

The ambition of this project is to set out a two-dimensional stellar evolution code, which fully takes into account the effects of rotation, at any rate and in a self-consistent way.

1. Espinosa Lara F. and Rieutord M. (2007), ``The dynamics of a fully radiative rapidly rotating star enclosed within a spherical box", in *Astron. Astrophys.*, vol. 470, p. 1013-1022

2. Rieutord M. and Espinosa Lara F. (2009), ``On the dynamics of a radiative rapidly rotating star", in *Comm. Asterosismology*, vol. 158., pp.˜99-103

3. Rieutord M. and Espinosa Lara F. (2013), ``Ab initio modelling of steady rotating stars", in *Seismology for studies of stellar rotation and convection*, Edts Goupil et al., *Lecture Notes in Physics*, vol. 865, p. 49, Springer

4. Espinosa Lara F. and Rieutord M. (2013), ``Self-consistent 2D models of fast rotating early-type stars", in *Astron. Astrophys.*, vol. 552, A35

5. Rieutord M., Espinosa Lara F. and Putigny B. (2016), ``An algorithm for computing the 2D structure of fast rotating stars", in *J. Computational Phys.*, vol. 318, 277-304

Formulation and numerical scheme

Espinosa et al.(2013)

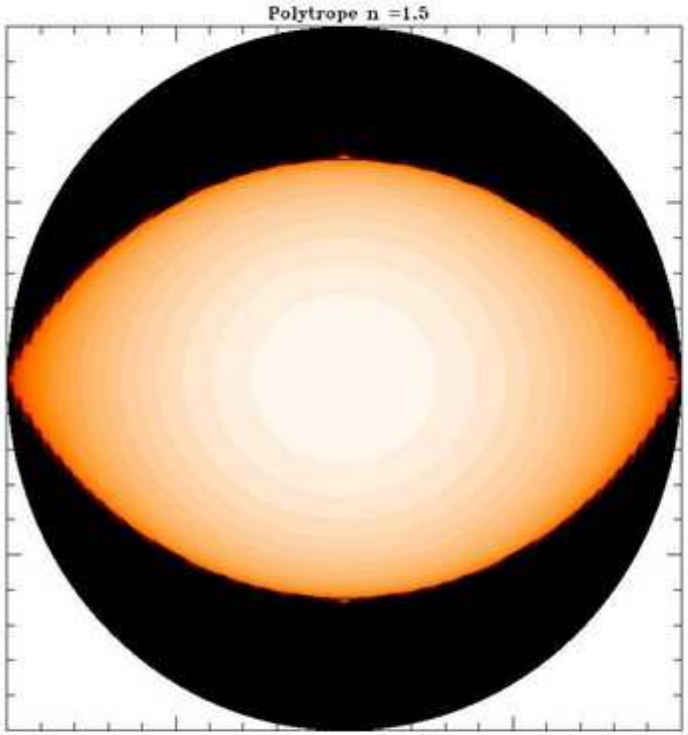
- 2D steady and axisymmetric stellar model based on multi-domain spectral method

$$\begin{cases} \Delta\phi = 4\pi G\rho \\ \rho T \boldsymbol{v}\cdot\boldsymbol{\nabla} S = -\boldsymbol{\nabla}\cdot\boldsymbol{F} + \varepsilon_* \\ \rho\boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v} = -\boldsymbol{\nabla}P - \rho\boldsymbol{\nabla}\phi + \boldsymbol{F}_v \\ \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) = 0. \end{cases}$$

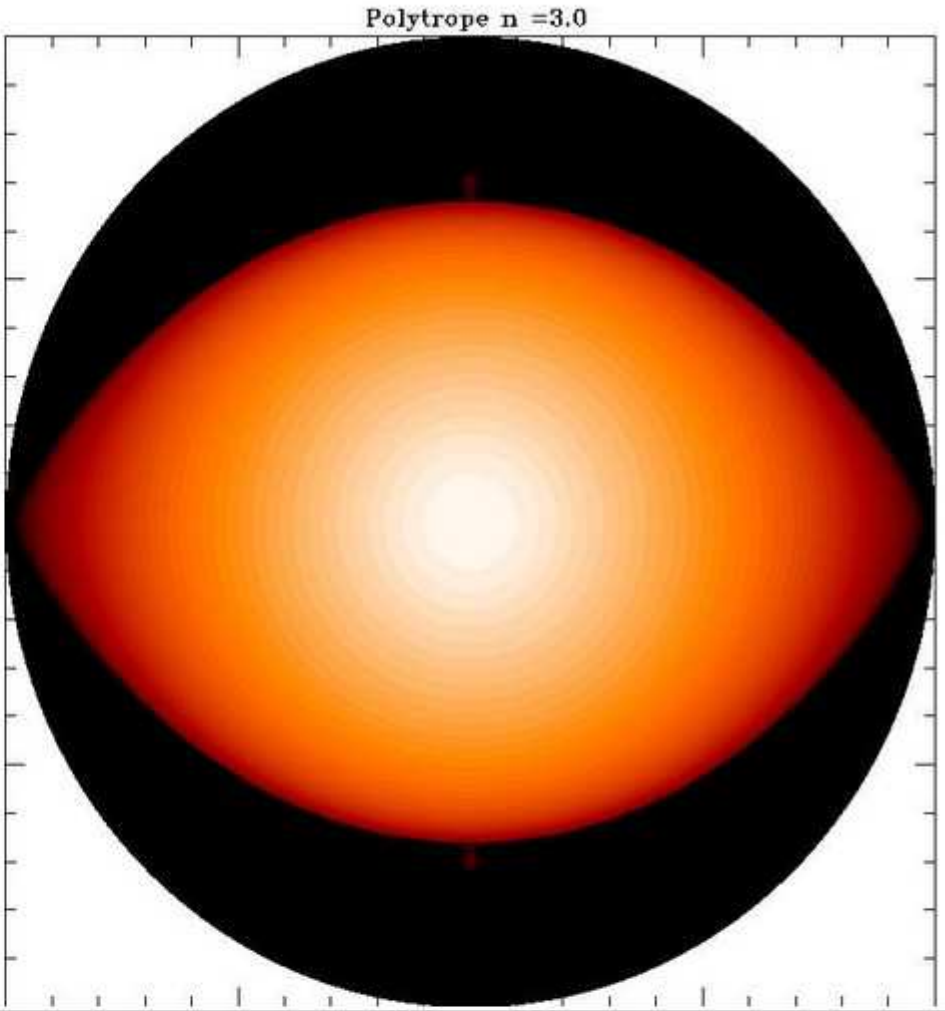
$$\boldsymbol{u} = \sum_{l=0}^{L_{\max}} Eu_l(r)\boldsymbol{R}_l^0 + Ev_l(r)\boldsymbol{S}_l^0 + w_l(r)\boldsymbol{T}_l^0,$$

- Self gravity
- Differential rotation + Meridional flow
- Viscosity

$$\boldsymbol{F}_v = \mu\boldsymbol{\mathcal{F}}_\mu(\boldsymbol{v}) = \mu\left[\Delta\boldsymbol{v} + \frac{1}{3}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{v}) + 2(\boldsymbol{\nabla}\ln\mu\cdot\boldsymbol{\nabla})\boldsymbol{v} + \boldsymbol{\nabla}\ln\mu\times(\boldsymbol{\nabla}\times\boldsymbol{v}) - \frac{2}{3}(\boldsymbol{\nabla}\cdot\boldsymbol{v})\boldsymbol{\nabla}\ln\mu\right].$$

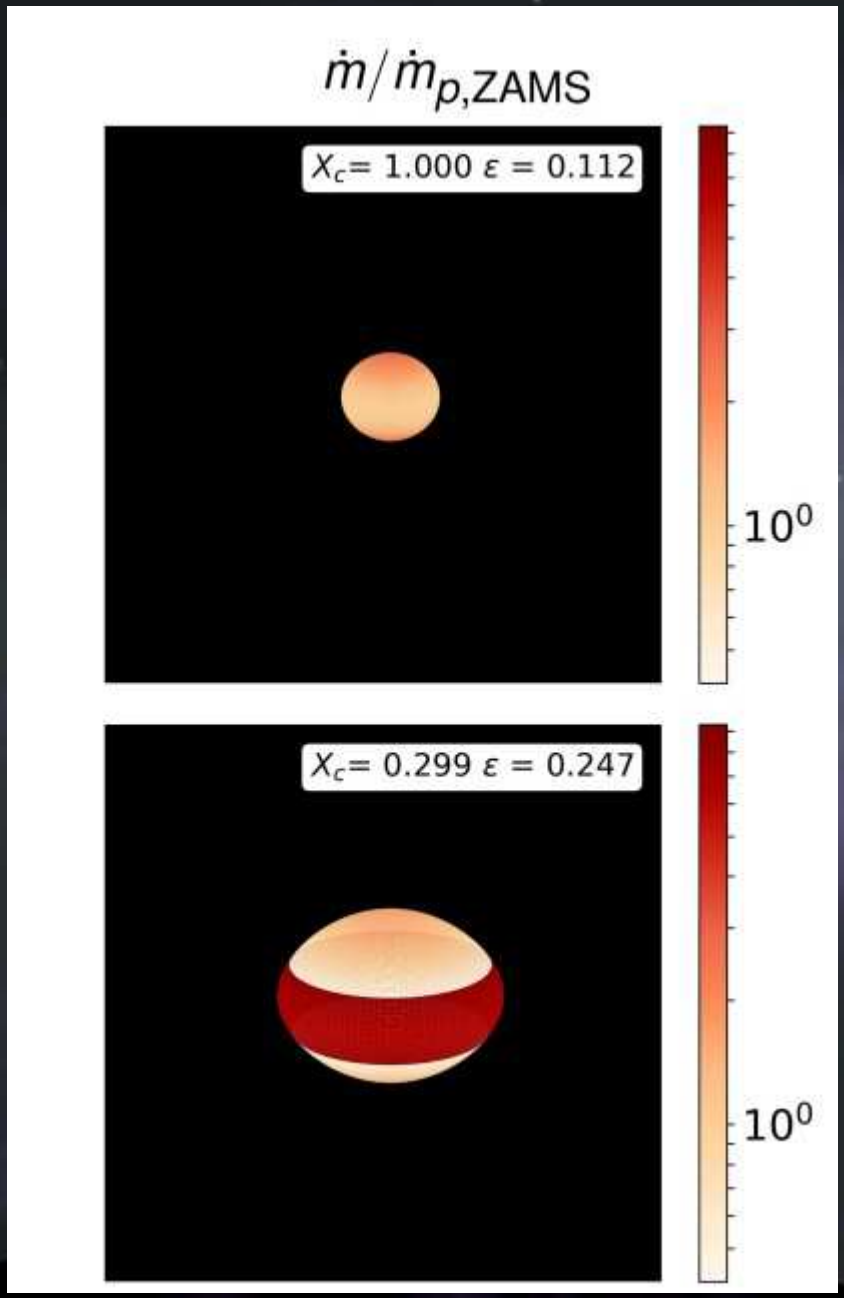


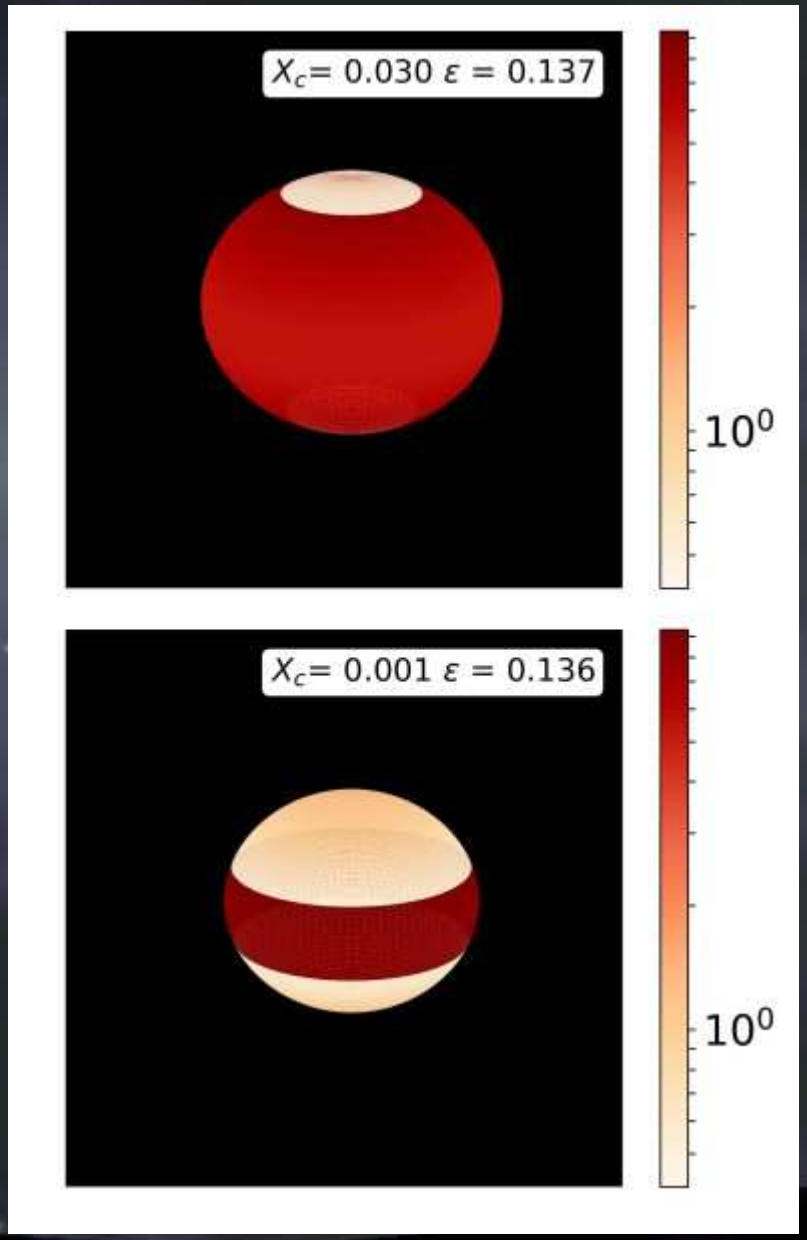
The shape of a n=3/2 polytrope rotating close to the break-up velocity.



The same for a n=3 polytrope; note that as this polytrope is more centrally condensed, its inner regions are less flattened.

Gagnier et al. (2019)





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- Weak solutions using triangle meshes
  (Yasutake, Fujisawa, Yamada 2015,2016)
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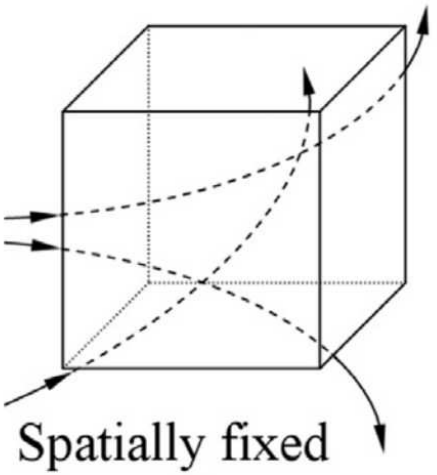
Shadloo et al. (2016)

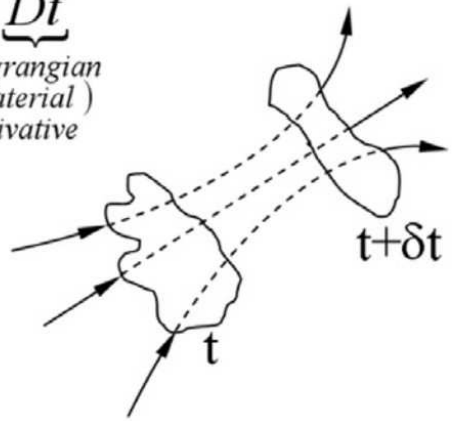
Eulerian

$$\underbrace{\frac{\partial}{\partial t}}_{\text{Eulerian derivative}} + \mathbf{v}\cdot\nabla = \underbrace{\frac{D}{Dt}}_{\text{Lagrangian (Material) derivative}}$$

Lagrangian



Spatially fixed volume element

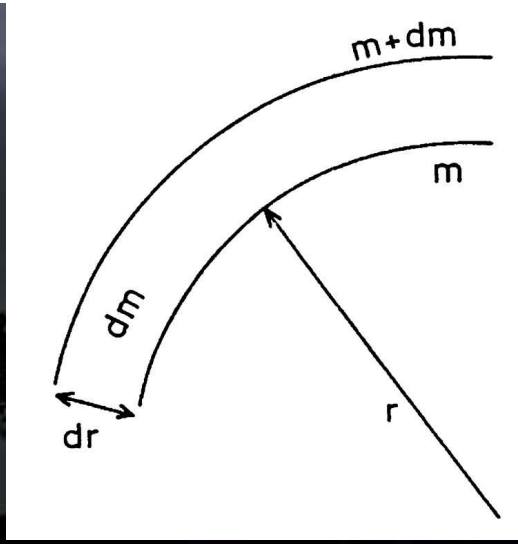


Following the motion of the fluid element

$$\frac{\partial P}{\partial r} = -\frac{Gm}{r^2}\varrho.$$

$$dm = 4\pi r^2 \varrho dr$$

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \varrho}, \qquad \frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^4}.$$



Rotating star on the Lagrangian coordinates

Calculate structures in Lagrangian

$$\frac{1}{\rho}\nabla p = -\nabla\phi_g + \Omega^2 R\boldsymbol{e}_R$$

$\Rightarrow$

$$\frac{1}{\hat{\rho}_{j,k}}\frac{P_{j+1,k}-P_{j,k}}{\hat{r}_{j+1}-\hat{r}_j} - F^{(T)r}_{j+\frac{1}{2},k} = 0,$$

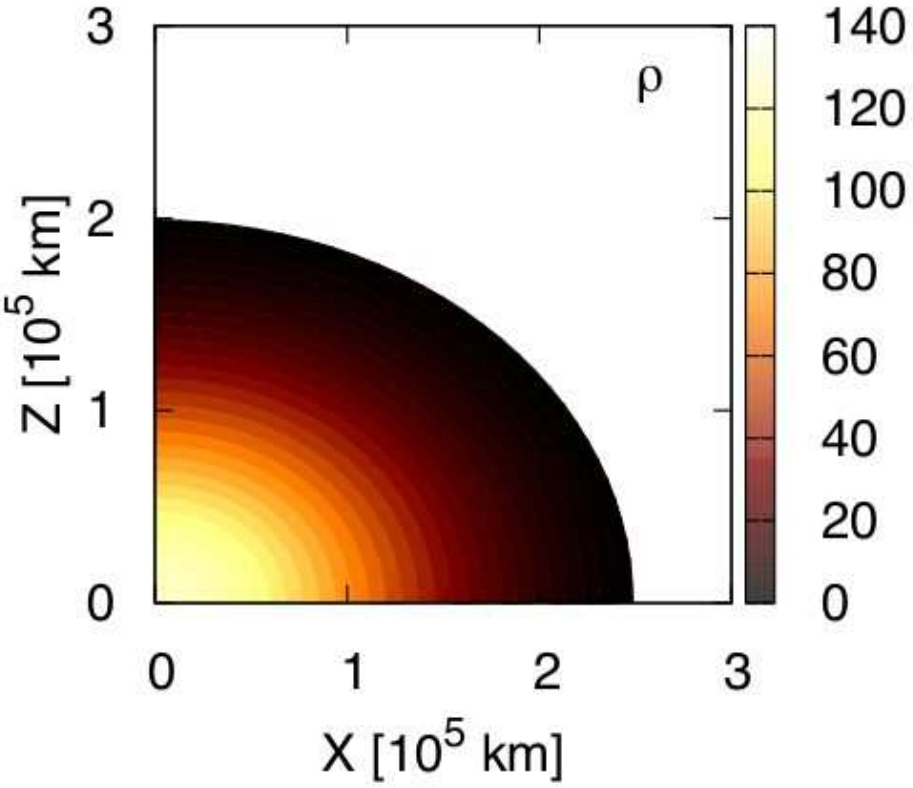
$$\frac{1}{\hat{\rho}_{j,k}\hat{r}_{j+\frac{1}{2},k}}\frac{P_{j,k+1}-P_{j,k}}{\hat{\theta}_{k+1}-\hat{\theta}_{k-1}} - F^{(T)r}_{j+\frac{1}{2},k} = 0,$$

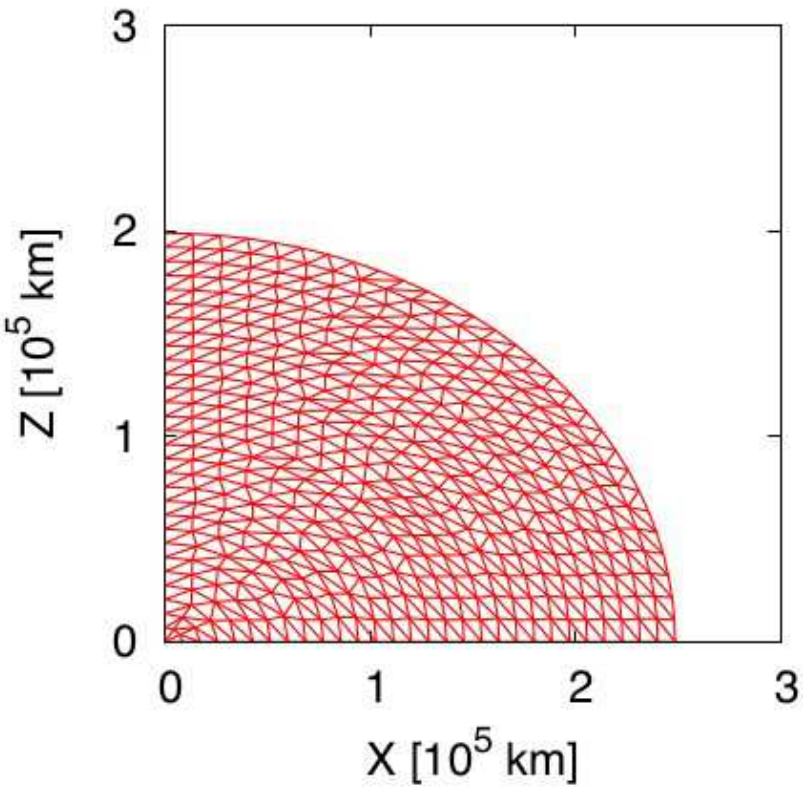
Solve “r” and “ θ ” instead of “ ρ ” and “p”!

Yasutake et al. (2015,2016)

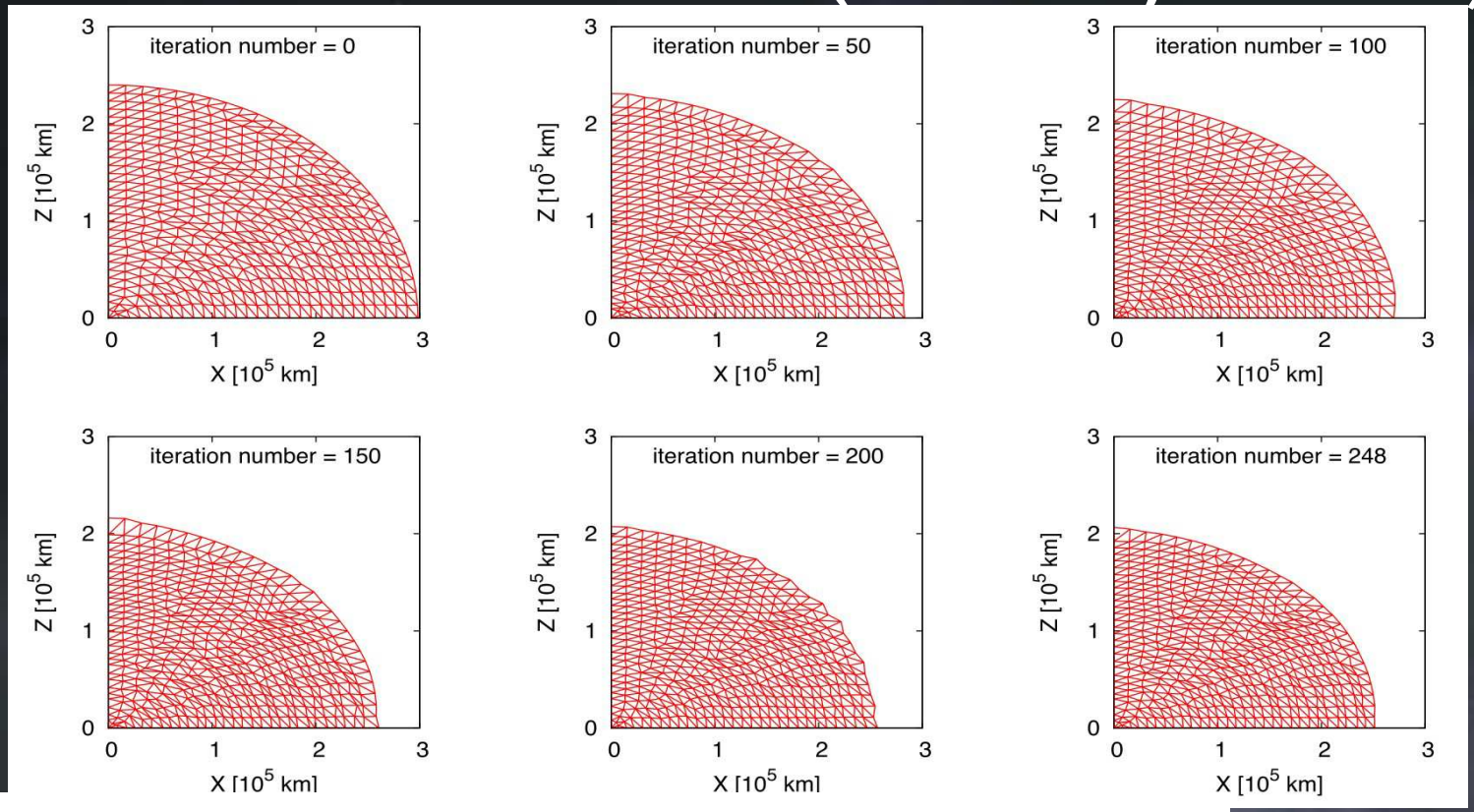
- Find energy minimum state (weak solution).

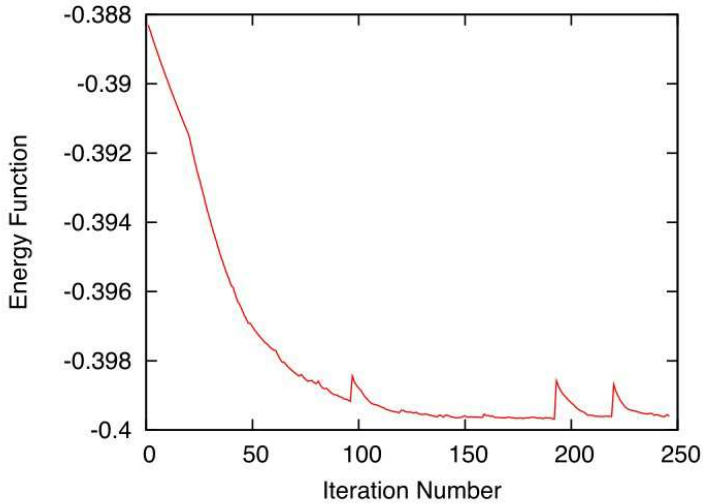
$$E(\xi) = \int \varepsilon\rho \, \mathrm{d}V + \frac{1}{2}\int \rho\phi \, \mathrm{d}V + \int \frac{1}{2}\rho\left(\frac{j}{\varpi}\right)^2 \mathrm{d}V.$$

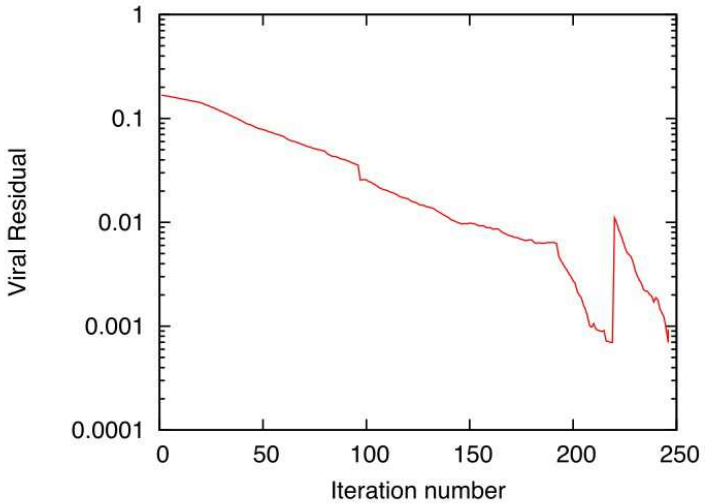




Yasutake et al. (2015,2016)







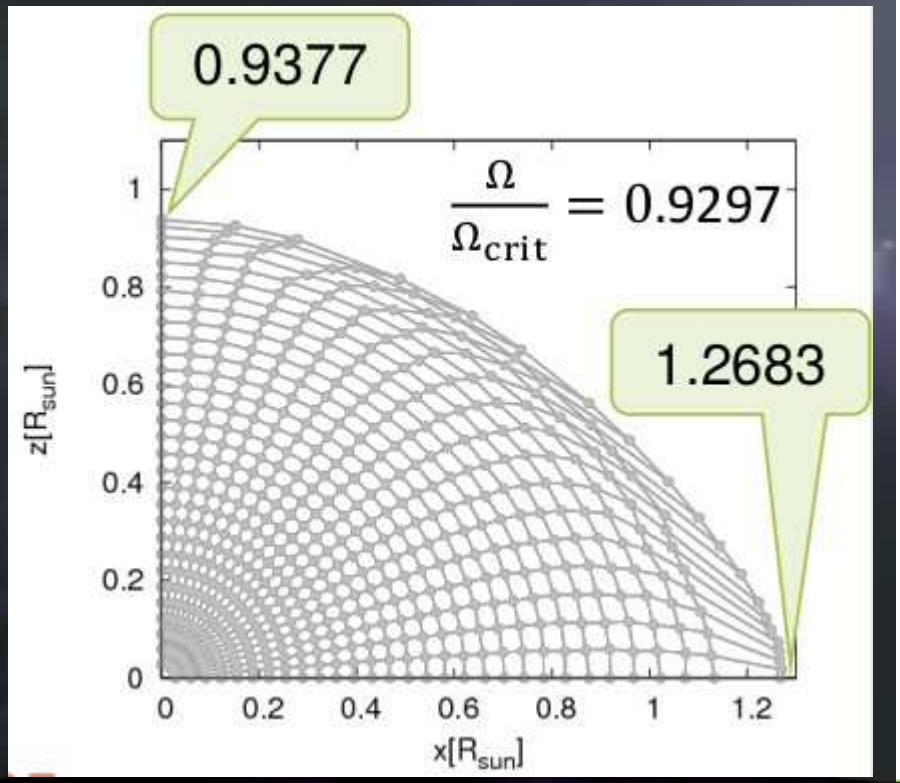
$$V_C = \left| \frac{2T + W + 3\int P \mathrm{d}V}{W} \right|$$

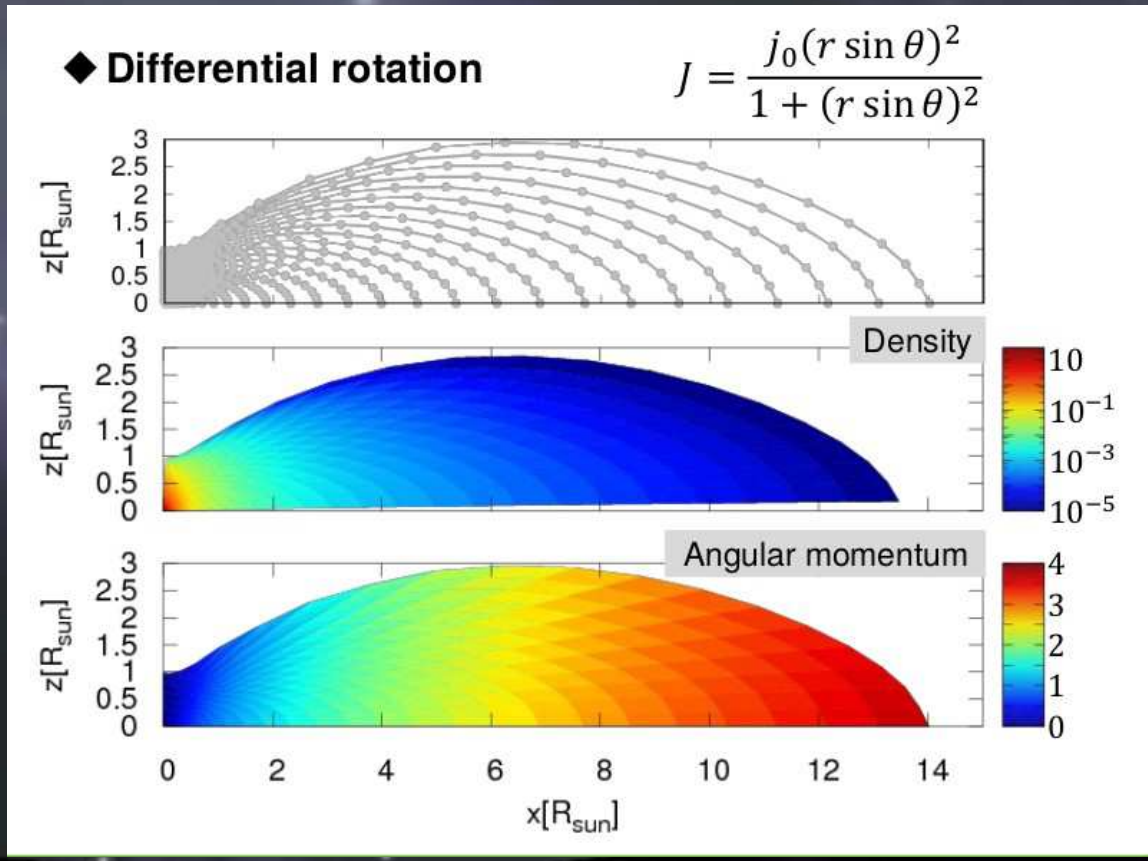
Finite difference method on the 2D Lagrangian coordinate

$$\frac{1}{\hat{\rho}_{j,k}}\frac{P_{j+1,k}-P_{j,k}}{\hat{r}_{j+1}-\hat{r}_j}-F^{(T)r}_{j+\frac{1}{2},k}=0,$$

$$\frac{1}{\hat{\rho}_{j,k}\hat{r}_{j+\frac{1}{2},k}}\frac{P_{j,k+1}-P_{j,k}}{\hat{\theta}_{k+1}-\hat{\theta}_{k-1}}-F^{(T)r}_{j+\frac{1}{2},k}=0,$$

Ogata et al. (see her poster)





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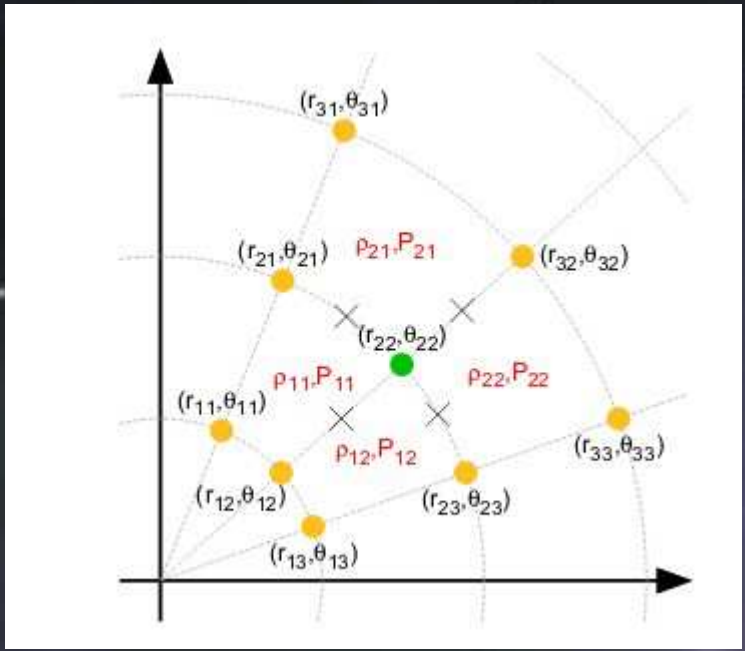
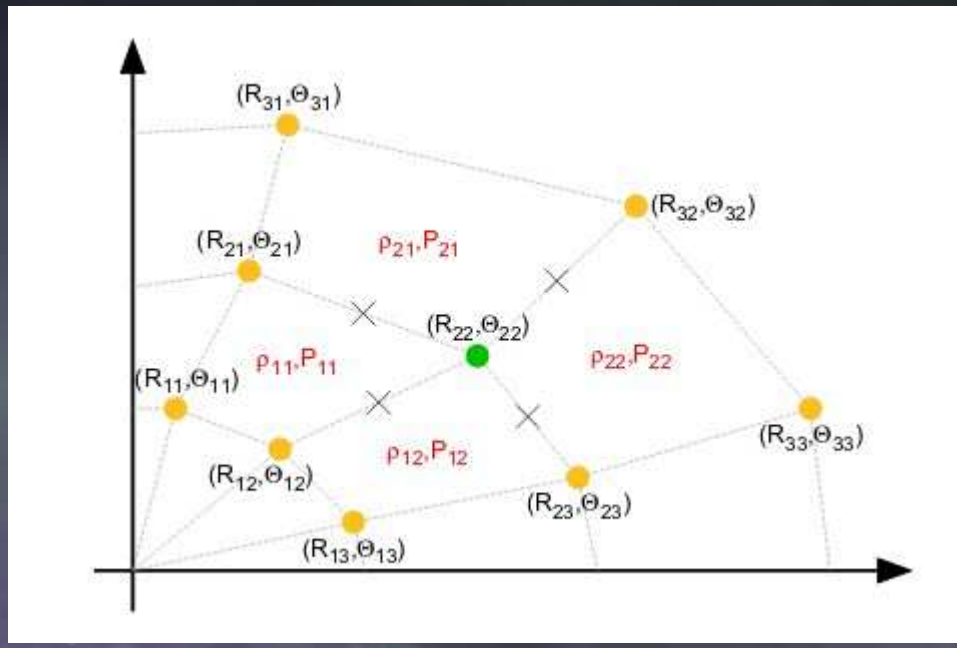
Spectral method on the 2D Lagrangian coordinates



$R(r,\theta)\Rightarrow$

$\Theta(r,\theta)\Rightarrow$



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Make non-rotating star as a reference model.

Consider the deformation due to the rotation as a map from the non-rotating star to the rotating star.

Expand R and Θ using orthogonal functions.

Numerical scheme

Non-linear Chebyshev-tau method

1. Using Chebyshev functions to expand the maps

$$R(r,\nu)=\sum_{n=0}^{N_{\max}}\sum_{\ell=0}^{L_{\max}}a_{n\ell}T_n(r)T_\ell(\nu)$$

$$\mu(r,\nu)=\sum_{n=0}^{N_{\max}}\sum_{\ell=0}^{L_{\max}}b_{n\ell}T_n(r)T_\ell(\nu).$$

$$\begin{aligned}T_0(x)&=1\\T_1(x)&=x\\T_2(x)&=2x^2-1\\T_3(x)&=4x^3-3x\end{aligned}$$

2.Set J and calculate ρ using R and μ

3.Expand the Euler equation

$$\mu=\cos\Theta,\qquad \nu=\cos\theta$$

$$\Delta\phi=4\pi G\rho$$

$$\frac{1}{\rho}\frac{\partial p}{\partial R}=-\frac{\partial\phi}{\partial R}+R\sin^2\Theta\Omega^2$$

$$p=K\rho^{1+\frac{1}{N}}$$

$$\int\int T_n(r)T_\ell(\nu)\left\{\frac{1}{\rho}\frac{\partial p}{\partial R}\right\}dr d\nu\Rightarrow F_{n\ell}$$

$$\int\int T_n(r)T_\ell(\nu)\left\{-\frac{\partial\phi}{\partial R}+R(1-\mu^2)\Omega^2\right\}dr d\nu\Rightarrow G_{n\ell}$$

$$F_{n\ell}-G_{n\ell}=0$$

- 4.Change coefficients (a,b) to reduce the residual iteratively.



Okawa, Fujisawa et al. Submitted.
ArXiv:1809.04495
Fujisawa, Okawa et al. (2019)
(see also our poster)

A new method for solving nonlinear eqs.

$$F_i(x_1, x_2, \cdots, x_N) = 0 \quad i = 1, 2, \cdots, N.$$

Newton-Raphson

$$\boldsymbol{x}^{n+1} = \boldsymbol{x}^n + \delta\boldsymbol{x},$$

$$\delta x_i = -\sum_{j=1}^{N} J_{ij}^{-1} F_j,$$

W4

$$\frac{d^2\boldsymbol{x}}{d\tau^2} + 2\frac{d\boldsymbol{x}}{d\tau} + M\boldsymbol{F} = 0,$$

$$\boldsymbol{x}^{n+1} = \boldsymbol{x} + \frac{1}{2}L^{-1}\boldsymbol{p}^n, \quad \boldsymbol{p}^{n+1} = \frac{1}{2}U^{-1}\boldsymbol{F}(\boldsymbol{x}$$

$$J \equiv UL \qquad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \ell_{21} & 1 & 0 & 0 \\ \ell_{31} & \ell_{42} & 1 & 0 \\ \ell_{41} & \ell_{42} & \ell_{43} & 1 \end{pmatrix} \qquad U = \begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{pmatrix}$$

Numerical scheme for nonlinear eqs.

- Newton-Raphson method is a most famous numerical method.
  - Initial guess is sufficiently close to the root,
    - Efficient
  - Initial guess is not close to the root,
    - Iteration usually unstable and we have problems with oscillations.

There are *no* good, general methods for solving systems of more than one nonlinear equations. Furthermore, it is not hard to see why (very likely) there *never will be* any good, general methods.

– Numerical recipe (Press, Teukolsky, et al.)

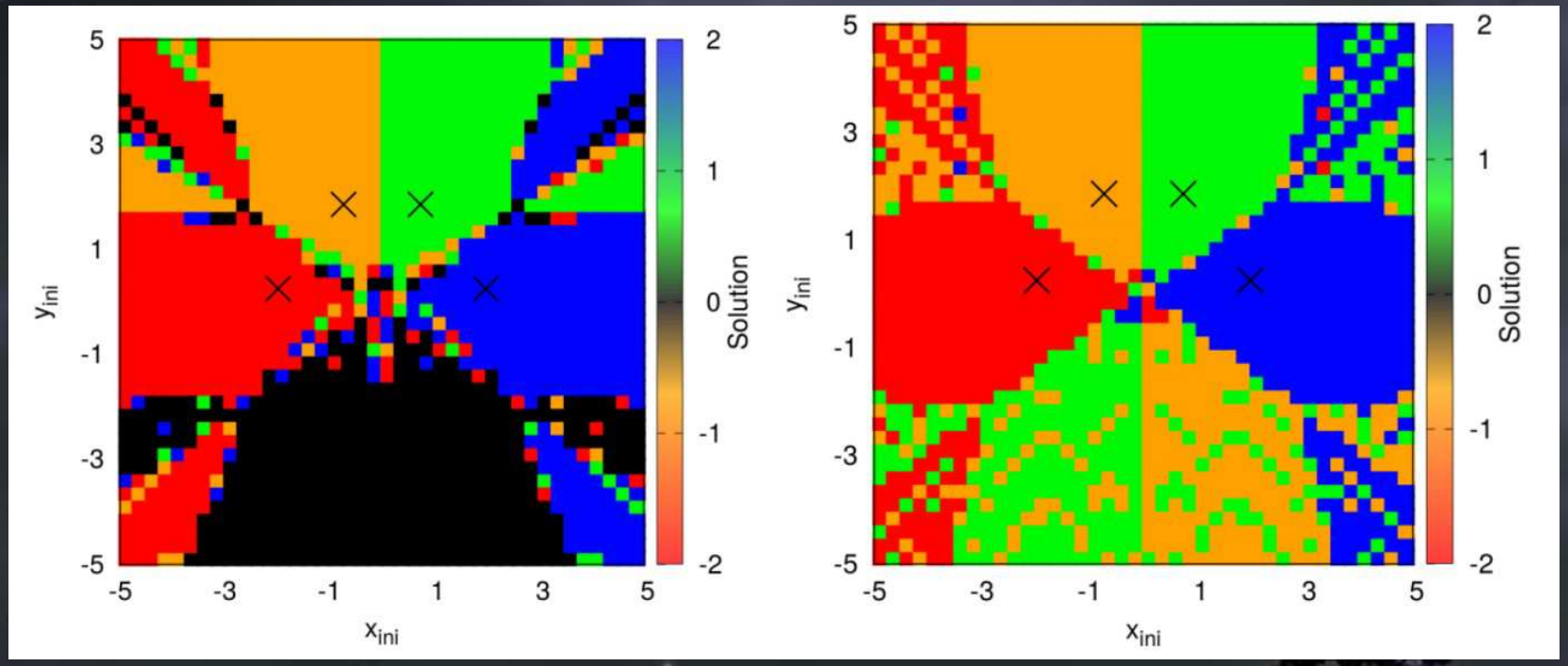
Convergence region (Basin)

See Okawa, Fujisawa's poster

$$f_1(x,y) = x^2 + y^2 - 4 = 0,\ f_2(x,y) = x^2 y - 1 = 0$$

NR

UL W4



Global convergent method!

Okawa, KF et al.
ArXiv:1809.04495

Numerical setup

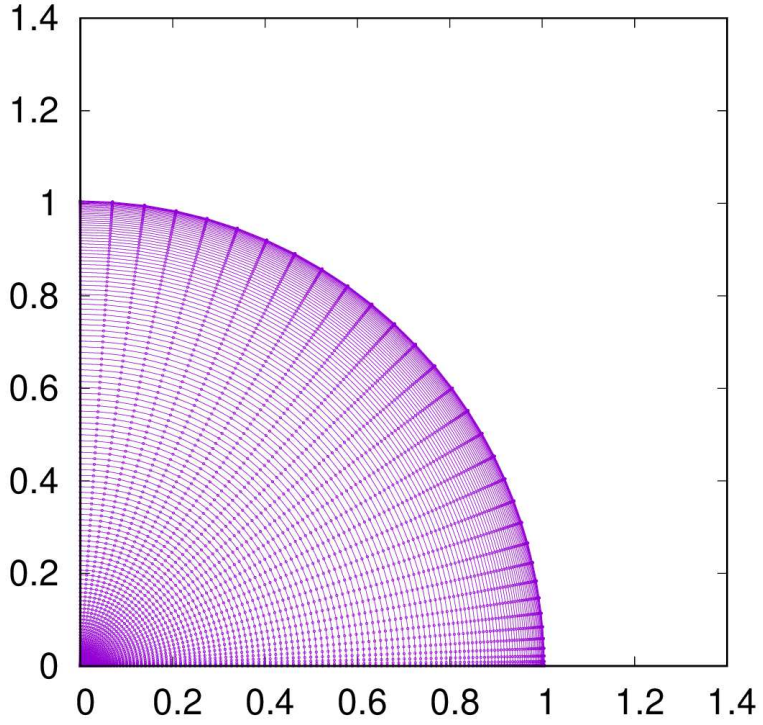
- Number of mass elements
  - 129 (r direction) x 33 (θ direction)
- EOS
  - Polytropic EOS $p = k\rho^{1+1/N}$
- Rigid rotation

  1. Spin-up solution sequence
     →Increase the total angular momentum
  2. Cooling (contraction) solution sequence
     →Decrease the entropy (K)

$$V_C = \left| \frac{2T + W + 3\int P dV}{W} \right| \sim 10^{-3}$$

Results (no rotation)

Mass elements



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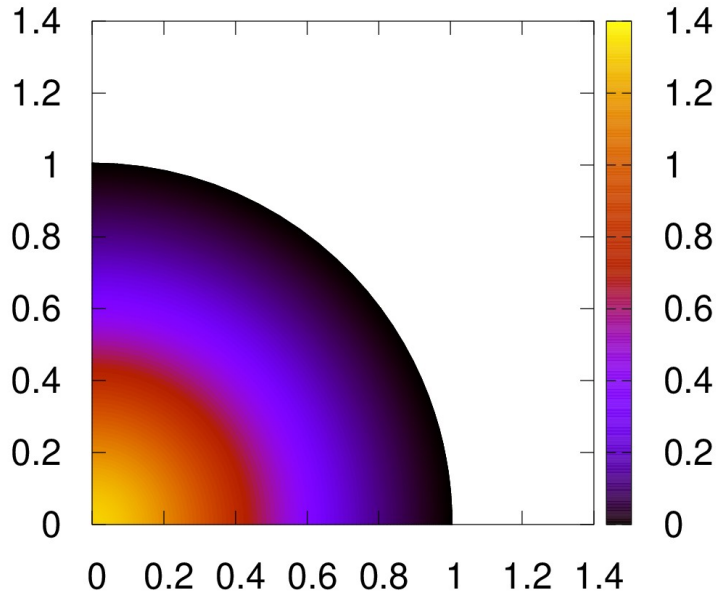
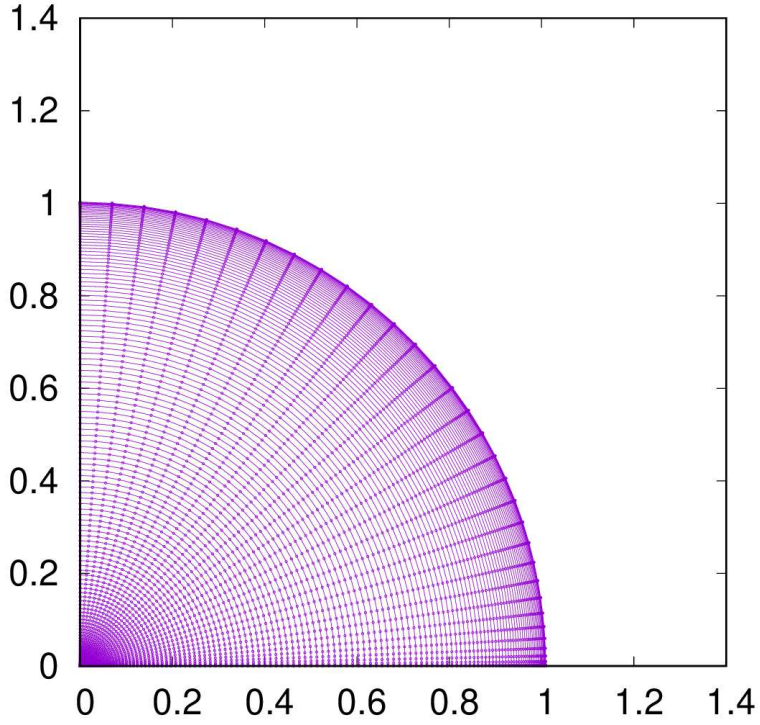


Results (spin-up)

Mass elements



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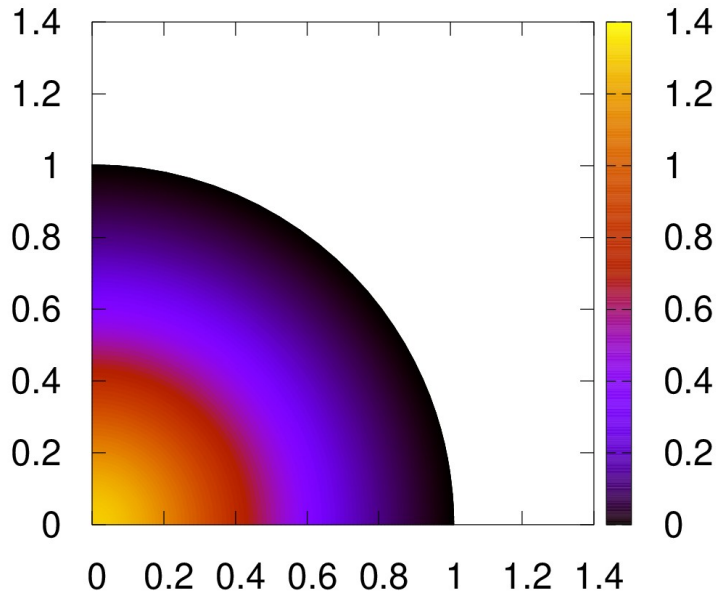
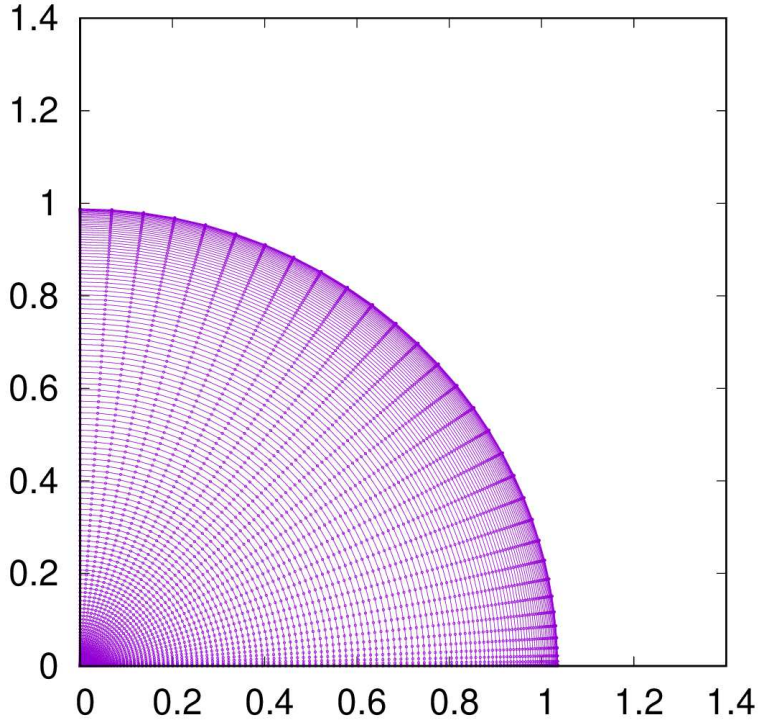


Results (spin-up)

Mass elements



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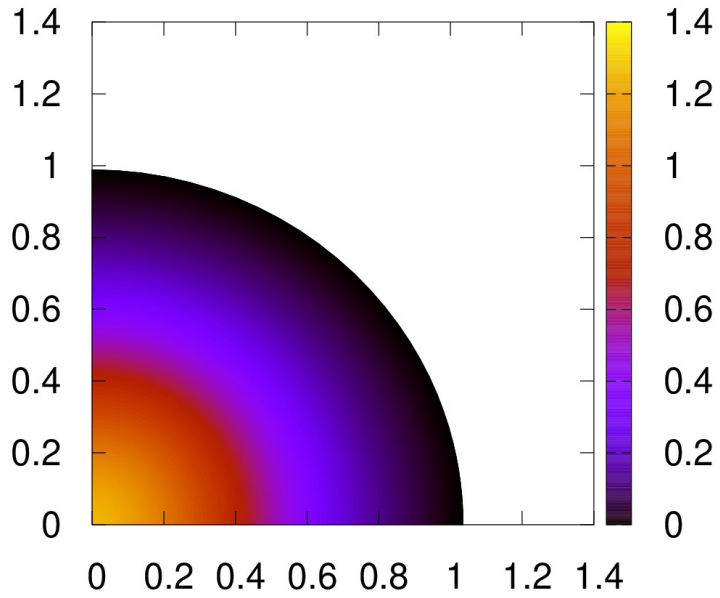
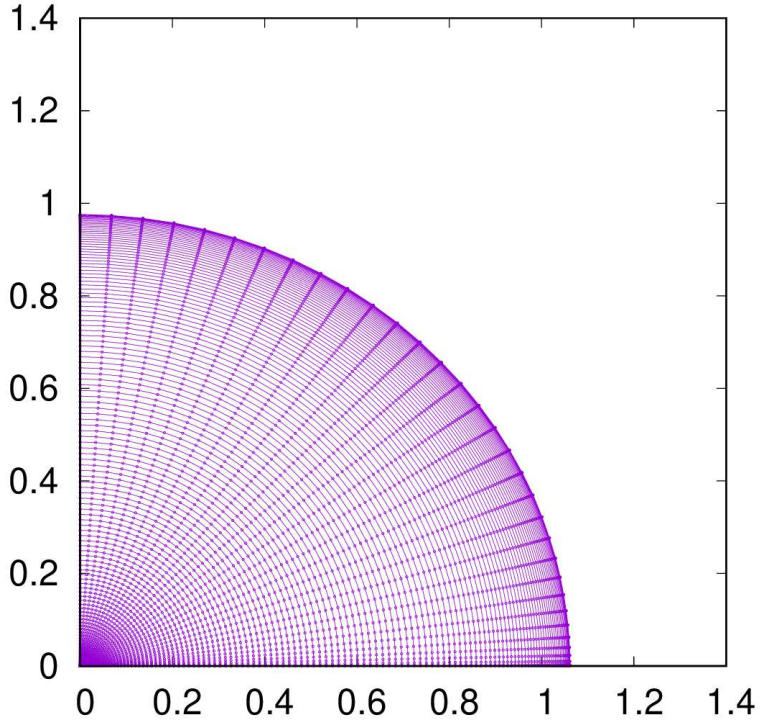


Results (spin-up)

Mass elements



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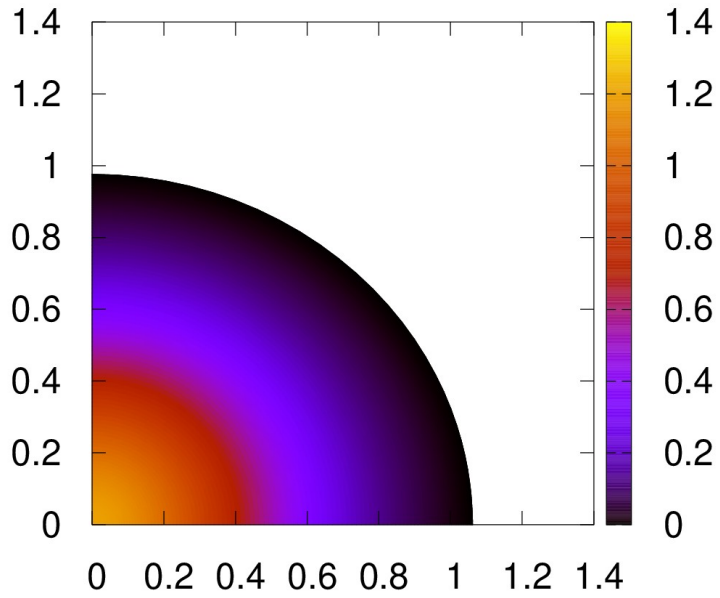
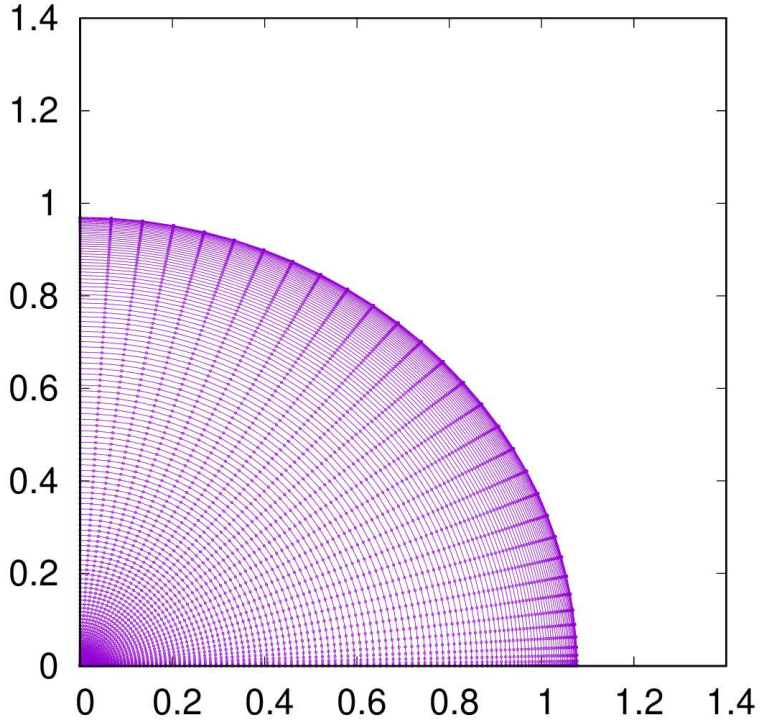


Results (spin-up)

Mass elements



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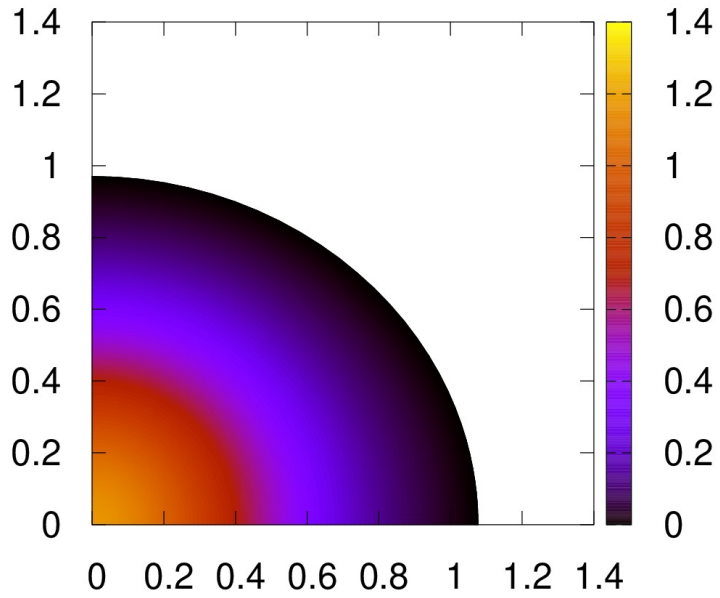
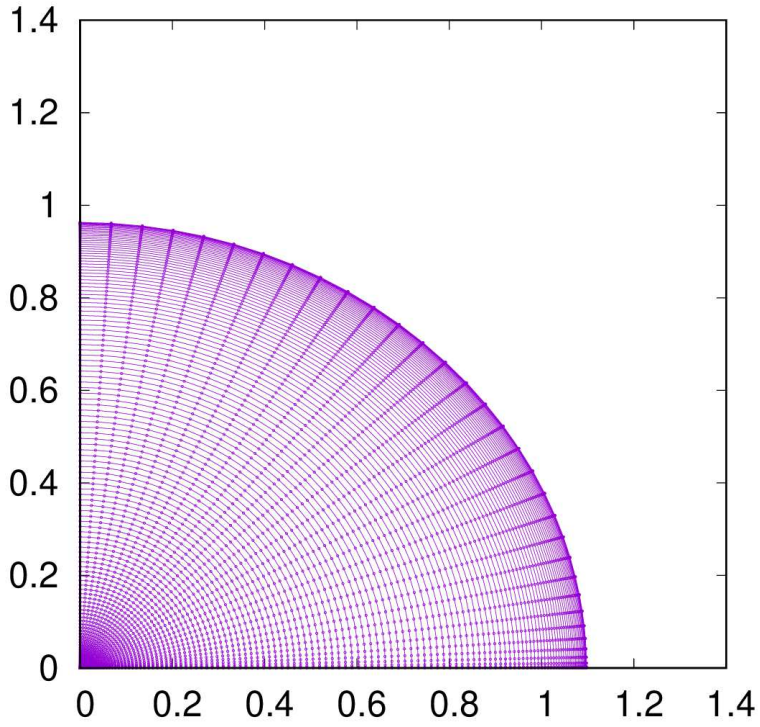


Results (spin-up)

Mass elements



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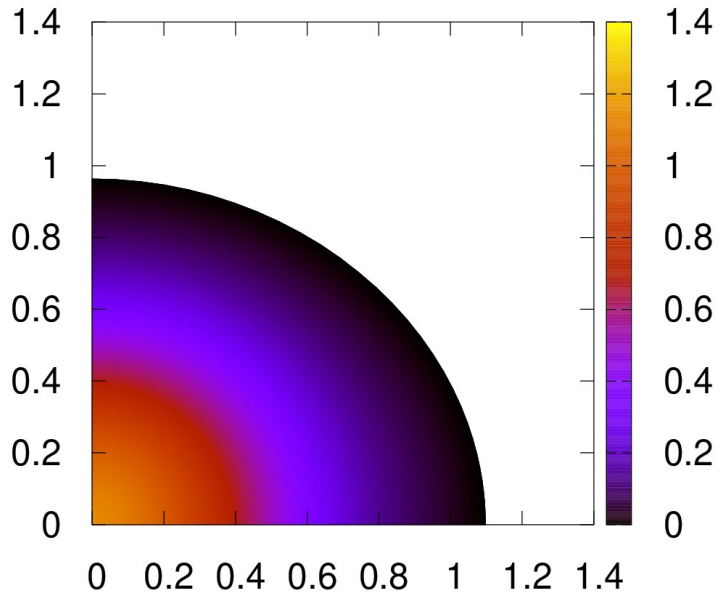
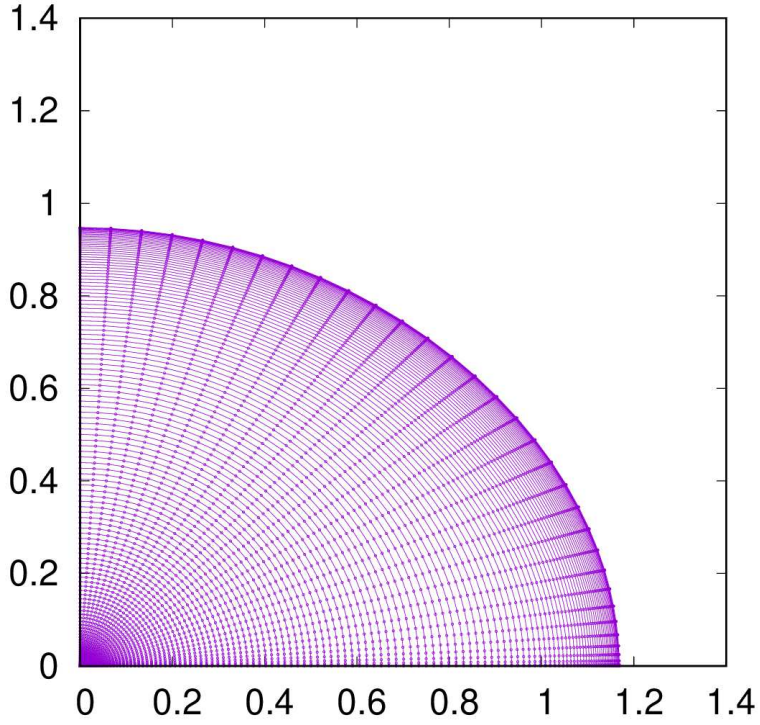


Results (spin-up)

Mass elements



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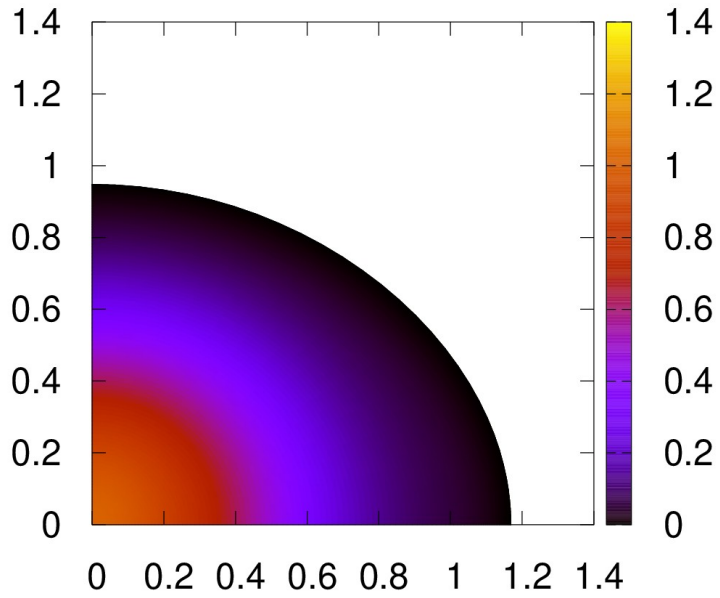
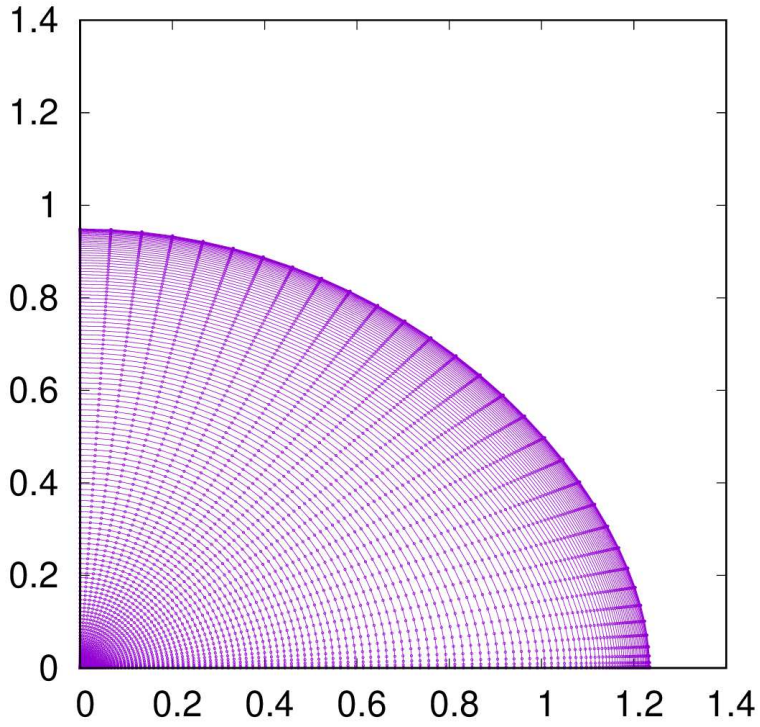


Results (spin-up)

Mass elements



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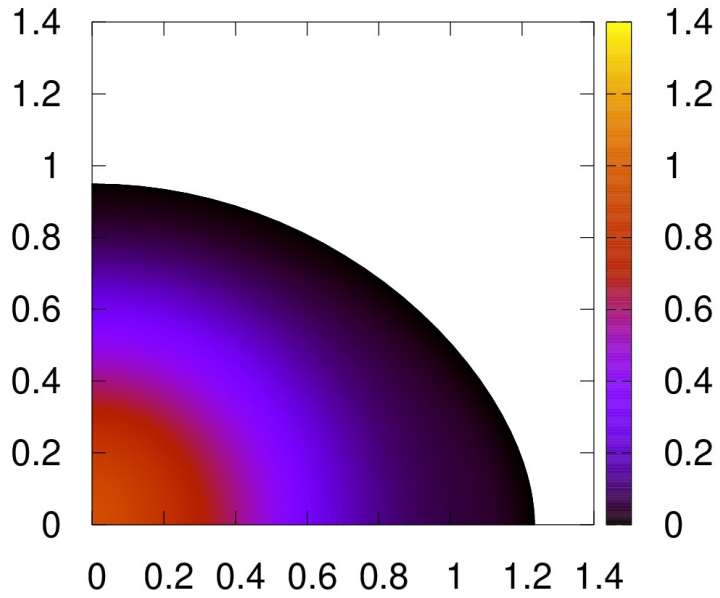

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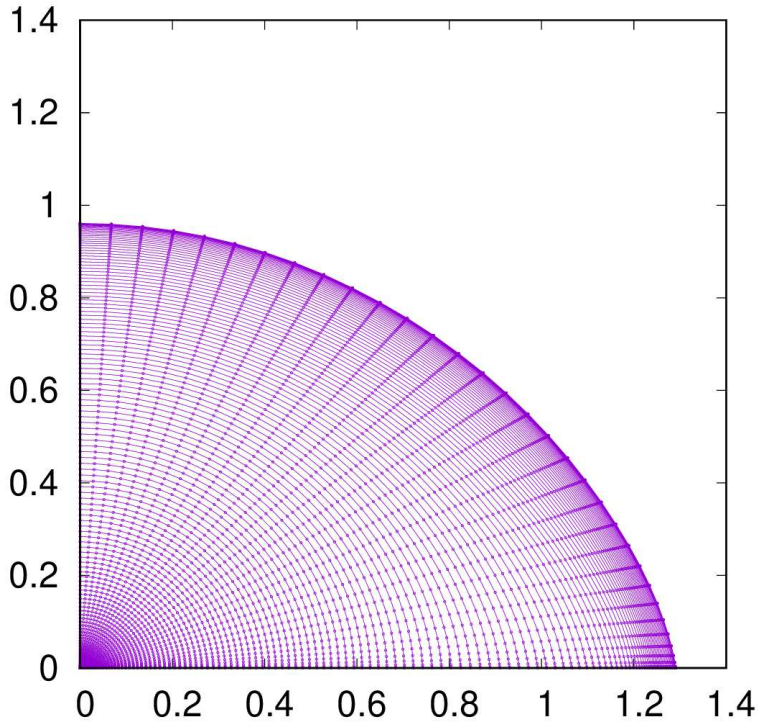
Mass elements



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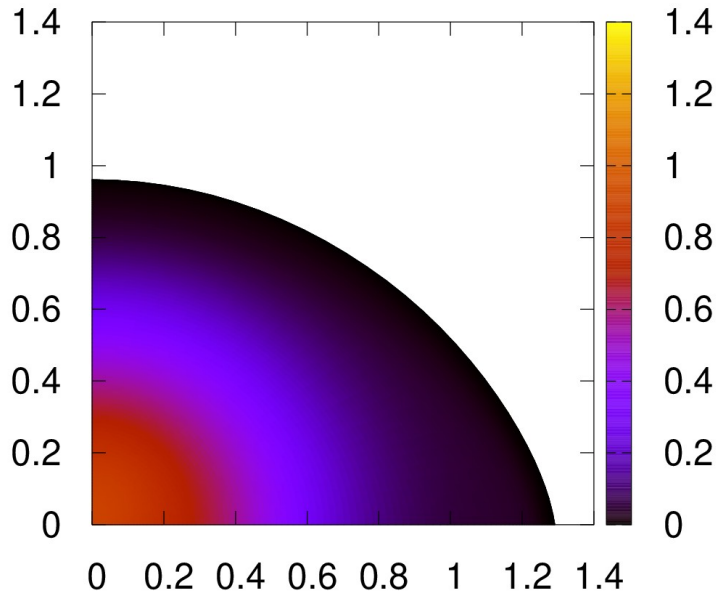
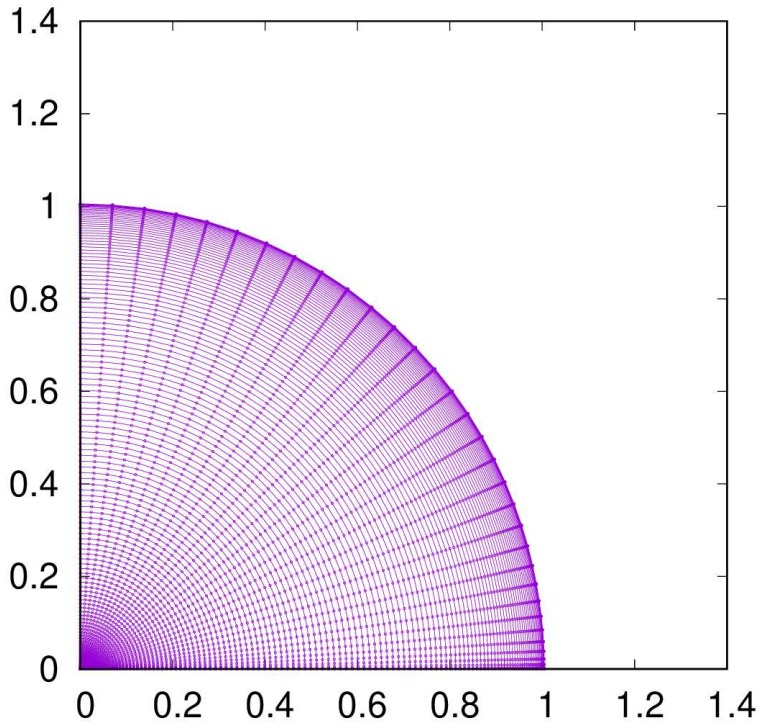
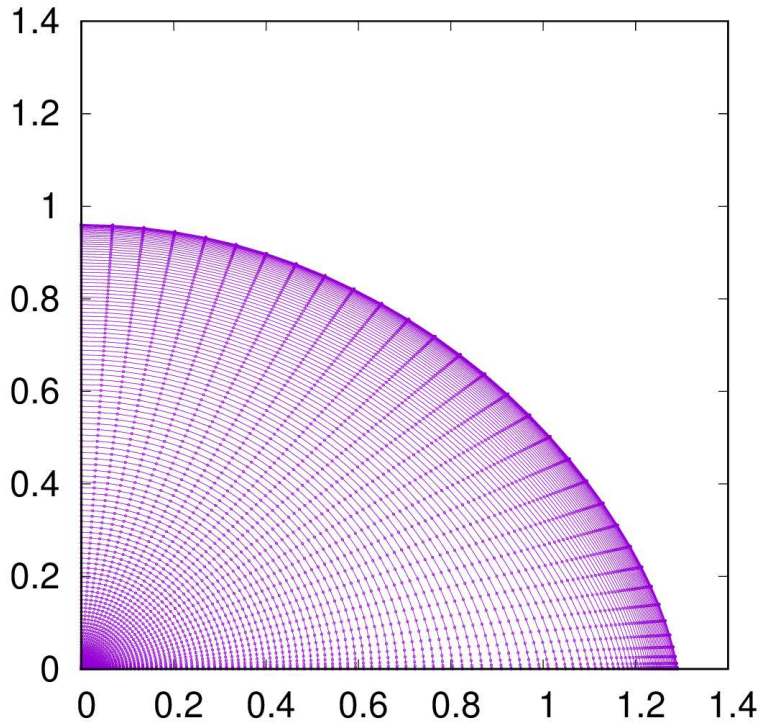


Results

No rotation

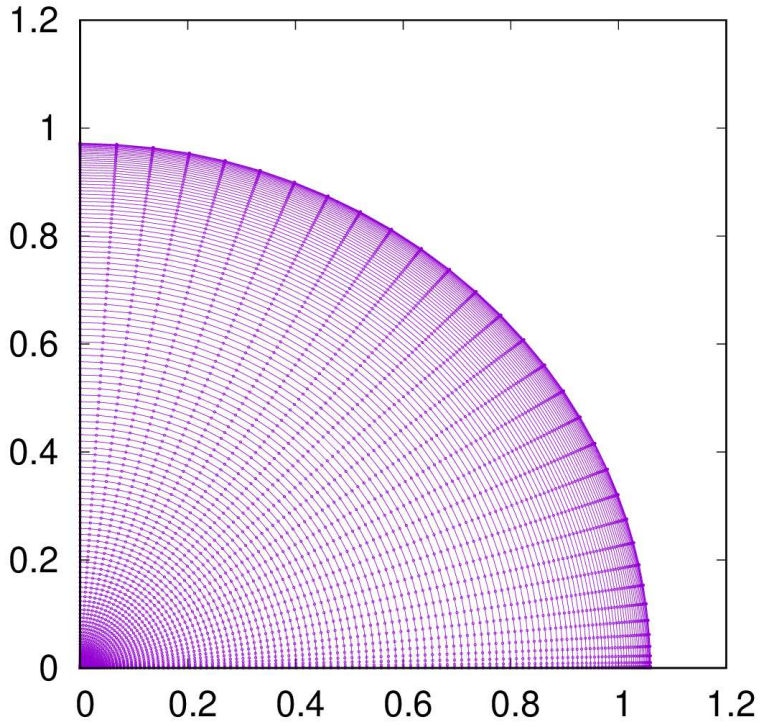


Rapid rotation



Results (cooling)

Mass elements



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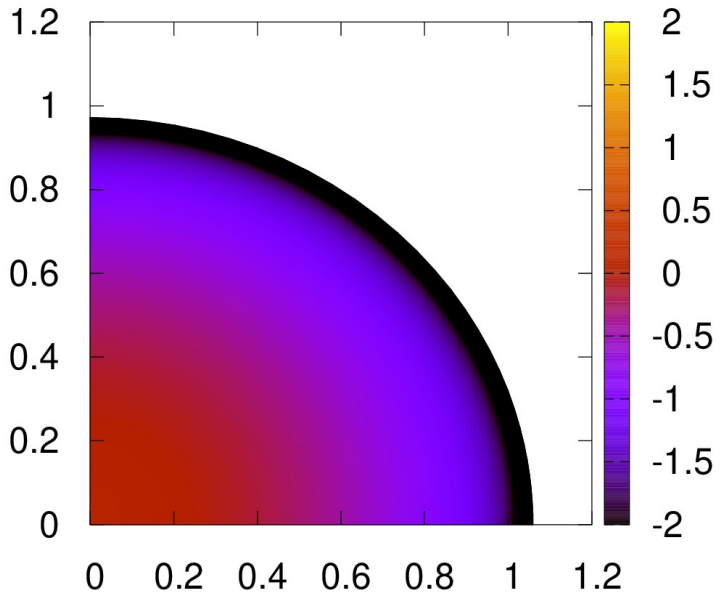

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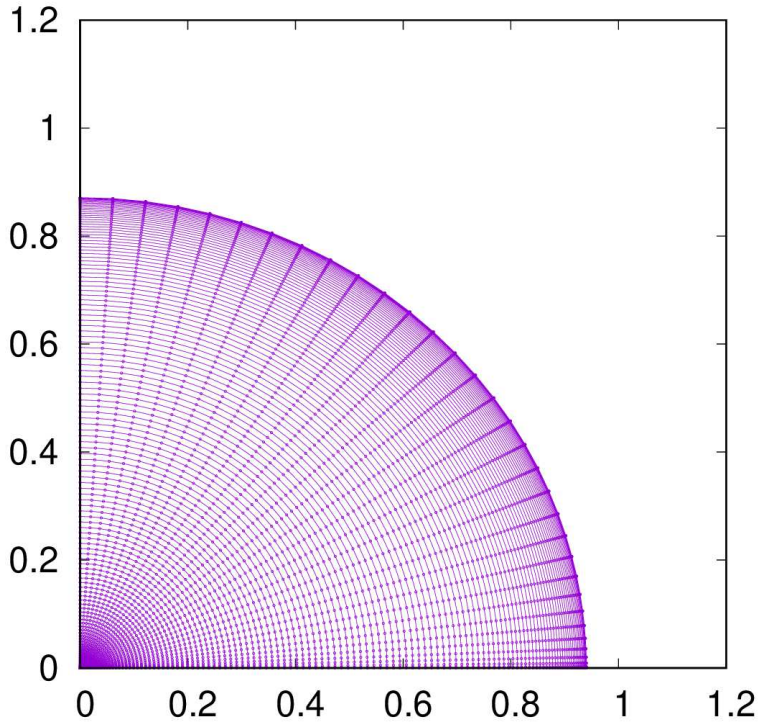

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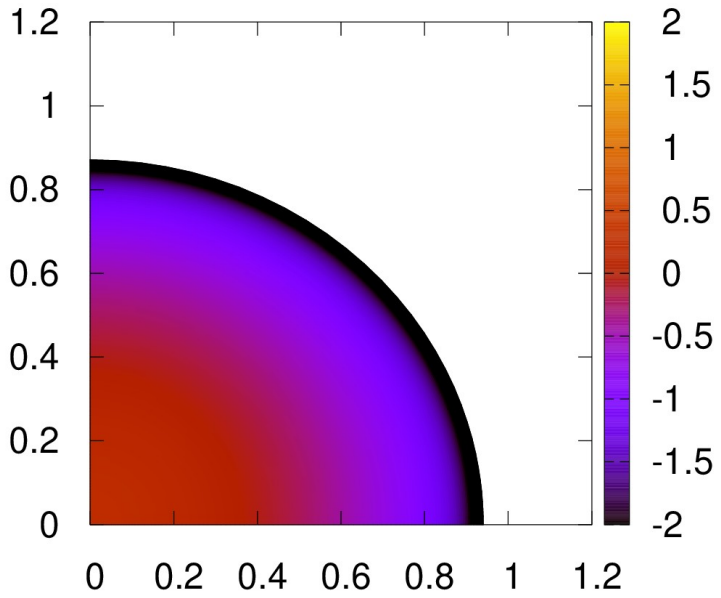

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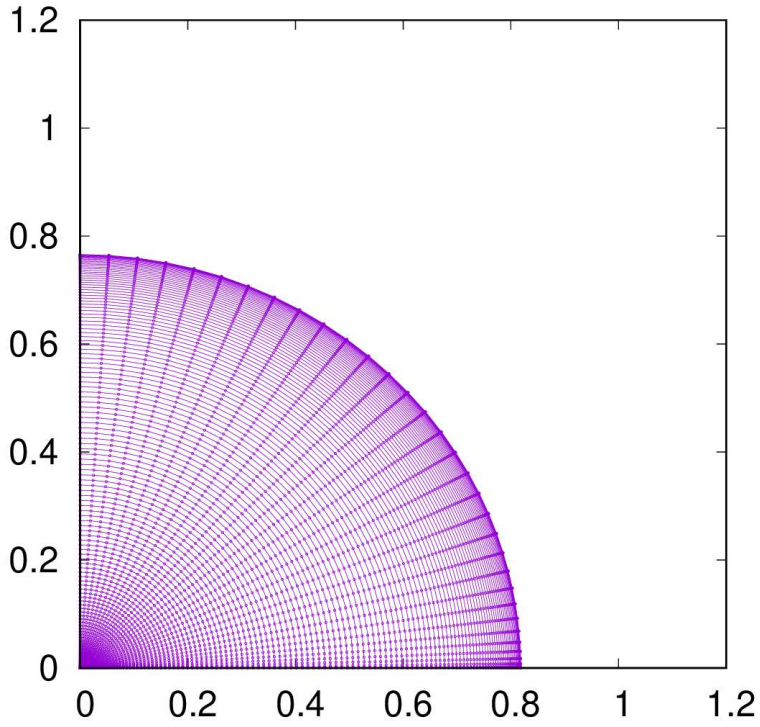

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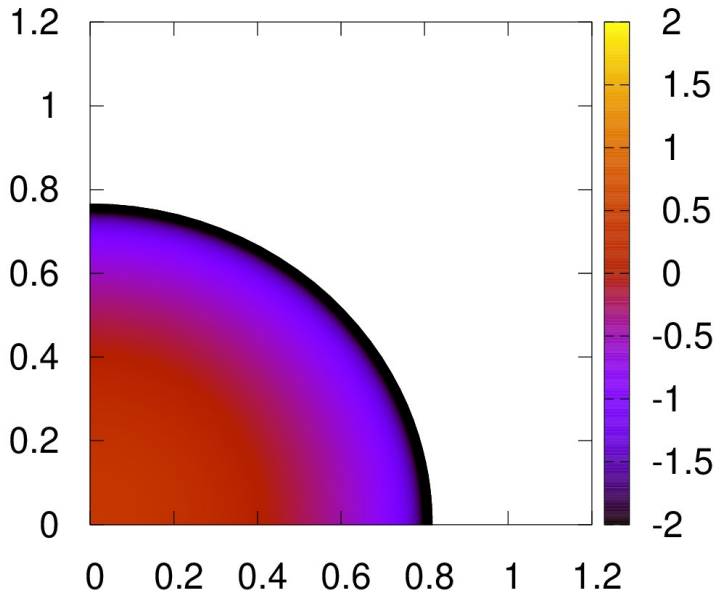

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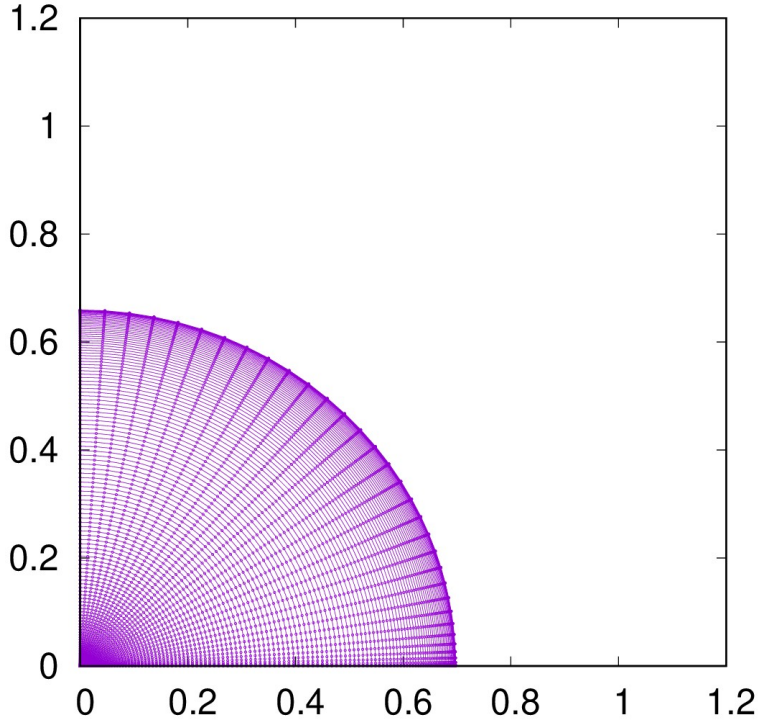
Mass elements



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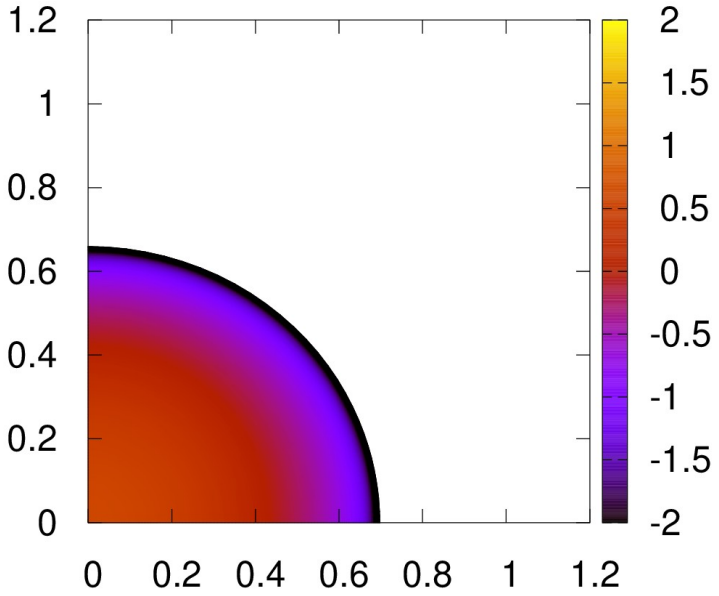

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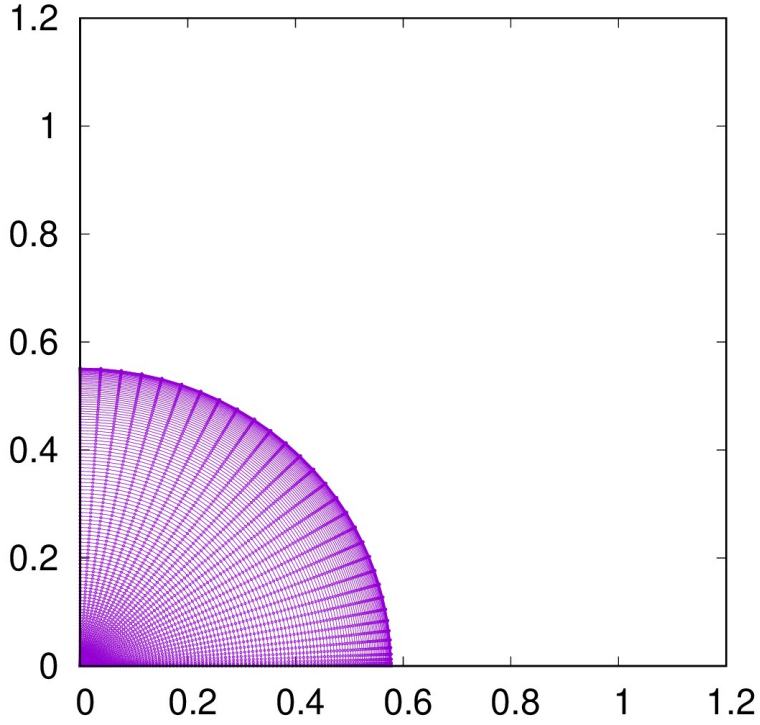

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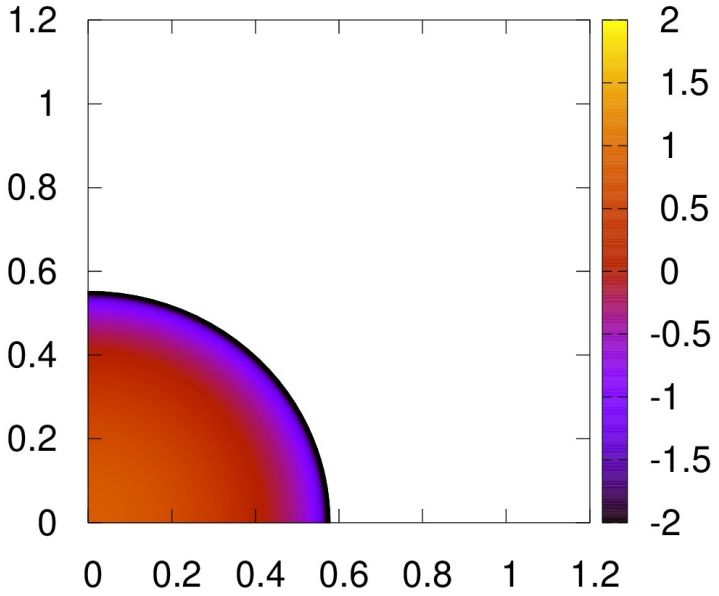

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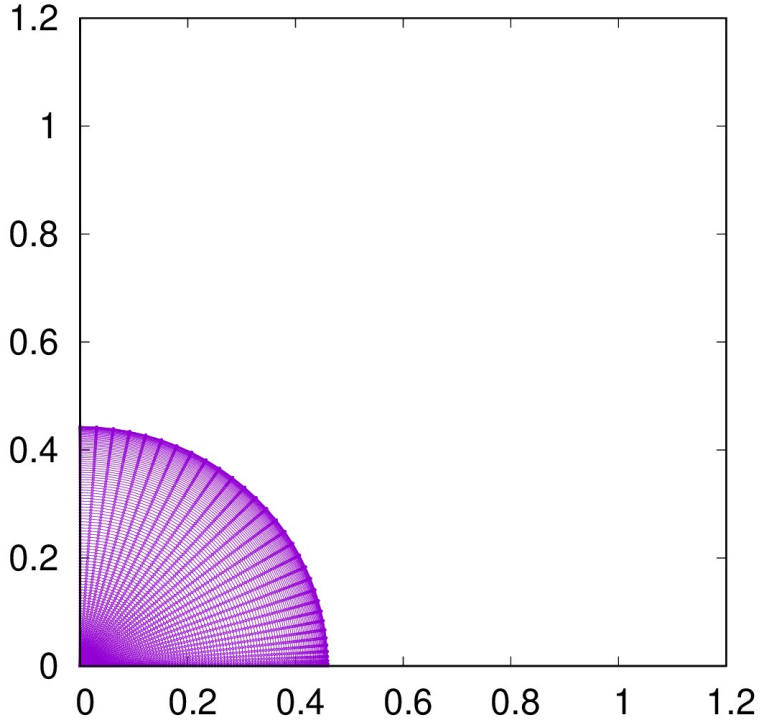
Mass elements



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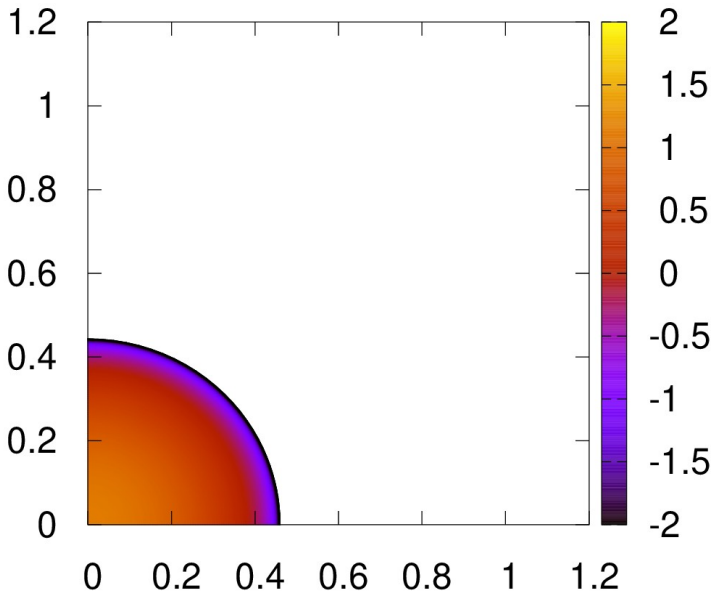

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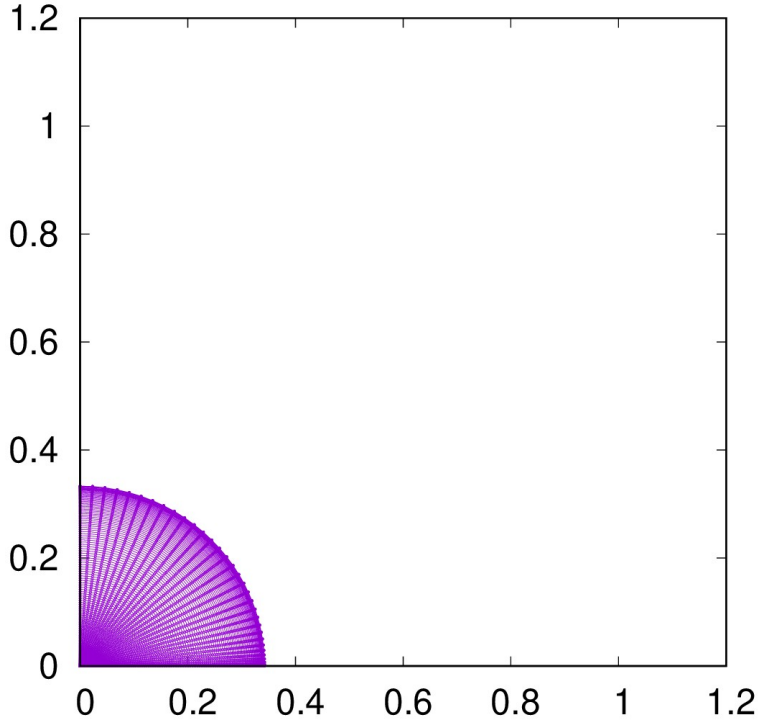
Mass elements



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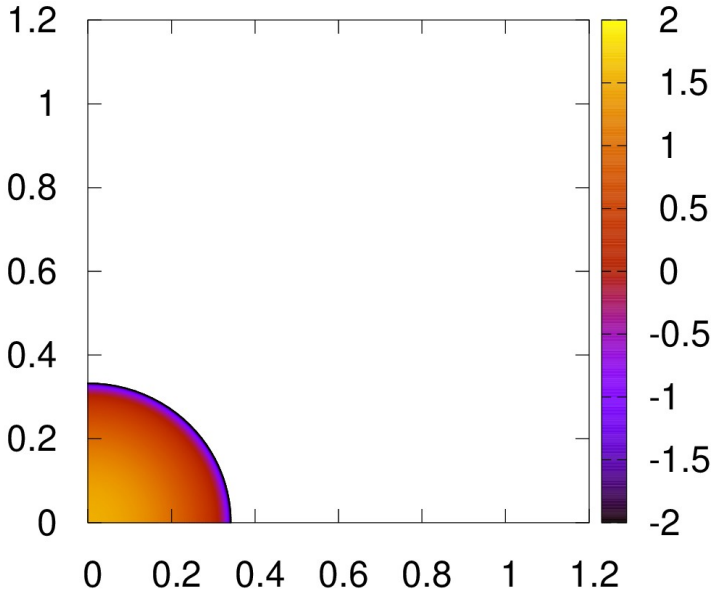

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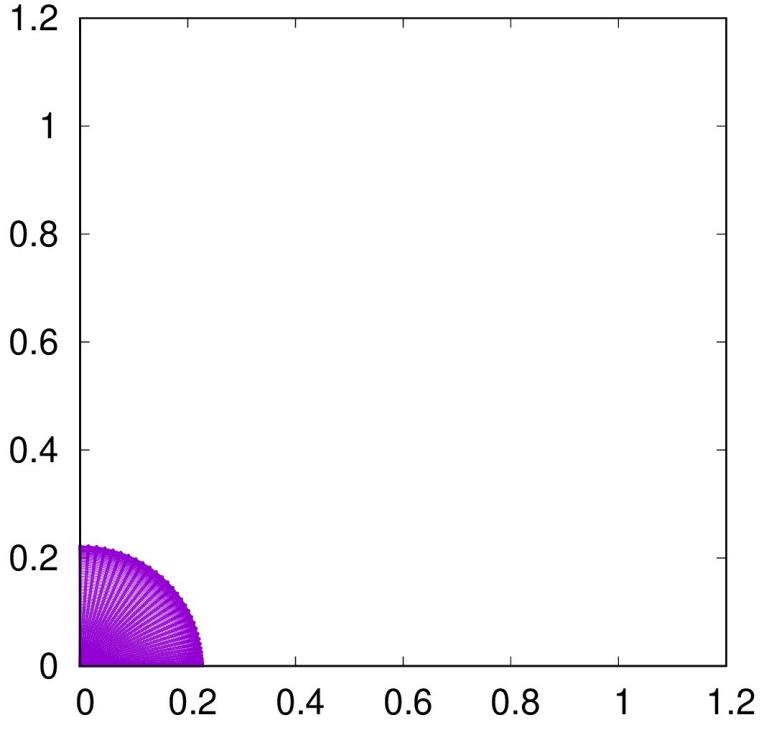
Mass elements



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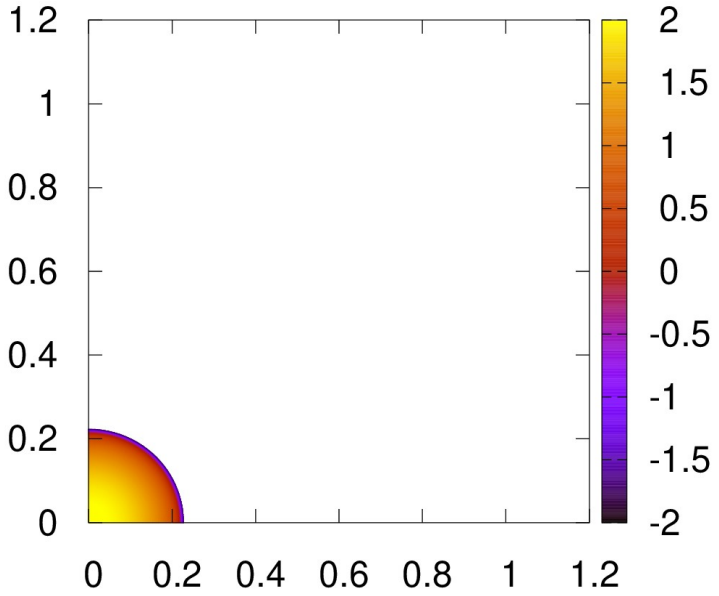

Summary and Future works

- Formulate rotating star on the 2D Lagrangian coordinates using spectral method
- Stable and accurate numerical scheme

- Stability analysis of the rotating star on Lagrangian coordinates
- 2D stellar evolution model

