The W4 method: a new multi-dimensional root-finding scheme for nonlinear systems of equations

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Abstract

We propose a new class of method for solving nonlinear systems of equations, which, among other things, has four nice features: (i) it is inspired by the mathematical property of damped oscillators, (ii) it can be regarded as a simple extention to the Newton-Raphson(NR) method, (iii) it has the same local convergence as the NR method does, (iv) it has a significantly wider convergence region or the global convergence than that of the NR method. In this article, we present the evidence of these properties, applying our new method to some examples and comparing it with the NR method.

1. Introduction

A system of nonlinear equations:

$$\boldsymbol{F}(\boldsymbol{x}) = 0, \qquad \text{where } \boldsymbol{x} \in \mathbb{R}^N,\ N \in \mathbb{Z} \text{ and } \boldsymbol{F} : \mathbb{R}^N \to \mathbb{R}^N$$

・We use a Newton-Raphson (NR) method widely, but we may fail to find solutions when the initial guess is far from the solution (Fig. 1).

Newton-Raphson (NR) and Damped Newton (DN) methods

$$F_1(x,y) := x^2 + y^2 - 4 = 0, \quad (1)$$
$$F_2(x,y) := x^2 y - 1 = 0.$$

$$J = \begin{bmatrix} 2x & 2y \\ 2xy & x^2 \end{bmatrix}.$$

$$\boldsymbol{x}_{n+1} = \boldsymbol{x}_n - \Delta\tau J^{-1}\boldsymbol{F}(\boldsymbol{x}_n), \quad (2)$$

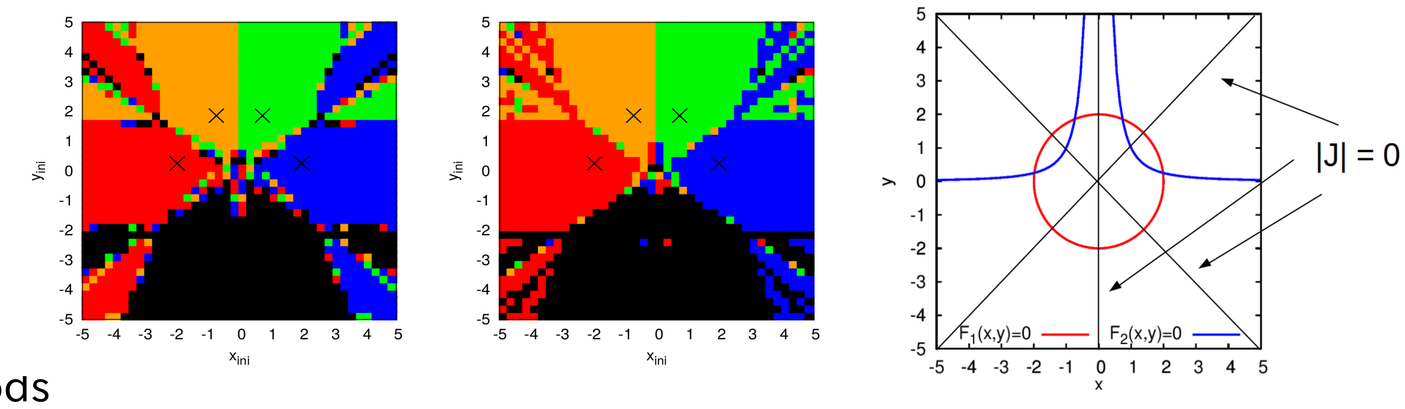


Fig.1 Initial guess dependence by the NR method with $\Delta\tau$ = 0.5 (Left) and the DN method(right) with $\Delta\tau$ = 0.5(Center). The red, orange, green and blue correspond to all solutions of eqution (2), while the black means the method cannot find any solution. In addition, four solutions are expressed by the black crosses. (Right) Contors of F1 = 0 (red) and F2 = 0 (blue). The domain is divided intro 6 regions by det |J| = 0 (black lines).

2. W4 method

・We regard equation (2) as a discretized form of the time-evolution equation:

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\tau} = -J^{-1}\boldsymbol{F}(\boldsymbol{x}). \quad \text{or more generally} \quad \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\tau} = -M\boldsymbol{F}(\boldsymbol{x}).$$

discretized ⇒ $\boldsymbol{x}_{n+1} = \boldsymbol{x}_n - \Delta\tau M\boldsymbol{F}(\boldsymbol{x}_n),$

・We consider the following time-evolution equation with the second derivative in time:

$$\frac{\mathrm{d}^2\boldsymbol{x}}{\mathrm{d}\tau^2} + M_1\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\tau} + M_2\boldsymbol{F}(\boldsymbol{x}) = 0, \qquad M_1, M_2 : \mathbb{R}^N \to \mathbb{R}^N.$$

divide into two equations ⇒

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\tau} = X\boldsymbol{p}, \qquad X\,Y \sim J^{-1}$$
$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}\tau} = -2\boldsymbol{p} - Y\boldsymbol{F},$$

$\boldsymbol{x} \in \mathbb{R}^N$ and $\boldsymbol{p} \in \mathbb{R}^N$ are generalized coordinates and associated momenta

discretized ⇒

$$\boldsymbol{x}_{n+1} = \boldsymbol{x}_n + \Delta\tau X\boldsymbol{p}_n,$$
$$\boldsymbol{p}_{n+1} = (1 - 2\Delta\tau)\,\boldsymbol{p}_n - \Delta\tau Y\boldsymbol{F}(\boldsymbol{x}_n).$$

Decomposition of J

1. UL decomposition (J = UL)

$X = L^{-1}$, $Y = U^{-1}$

U: Upper triangular matrix
L: Lower triangular matrix

2. LDH decomposition (J = LH) [2]

$X = H^{-1}$, $Y = L^{-1}$

H: Householder matrix ($H^T = H^{-1} = H$)
L: Lower triangular matrix

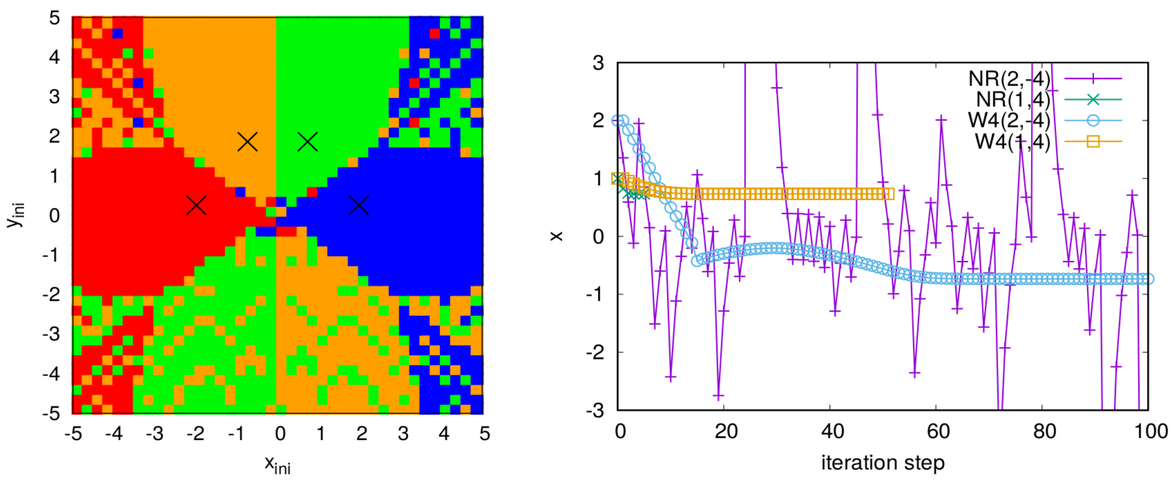


Fig. 2 (Left) the basin for the W4 method with $X = L^{-1}$, $Y = U^{-1}$ and $\Delta\tau$ = 0.5. The red, orange, greenand blue colors correspond to the initial conditions to reach one of the four solutions of Eq. (1) indicated with the black crosses as in Fig. 1. (Right) the sequences produced by the itetaion maps of the NR and W4 methods for the two initial conditions: $(x_0, y_0) = (1, 4)$ and $(2, -4)$.

3. Astrophysical applications

1D GR star with fixing mass [3]

- **Metric ansatz** : $\mathrm{d}s^2 = -f(r)^2\mathrm{d}t^2 + g(r)^2\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right)$
- **Einstein eqs.** $(E^\mu_\nu \equiv G^\mu_\nu - \kappa_G T^\mu_\nu)$:

$$r^2 E^0_0 = \kappa_G r^2\rho - \frac{2r}{g^3}\frac{\mathrm{d}g}{\mathrm{d}r} - 1 + g^{-2},$$
$$-r^2 E^1_1 = \kappa_G r^2 P - \frac{2r}{fg^2}\frac{\mathrm{d}f}{\mathrm{d}r} + 1 - g^{-2},$$

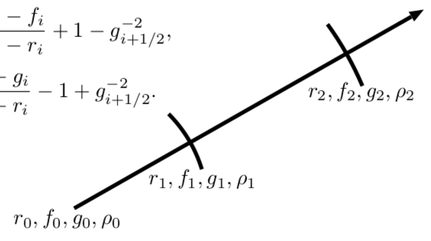
- **Conservation of the perfect fluid** : $\dfrac{1}{P+\rho}\dfrac{\mathrm{d}P}{\mathrm{d}r} + \dfrac{1}{f}\dfrac{\mathrm{d}f}{\mathrm{d}r} = 0,$
- **Equation of State** : e.g. $P = P(\rho) = k\rho^\Gamma,$

Discretized Eqs. :

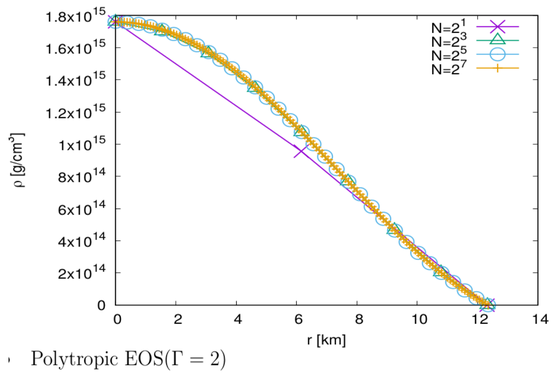
$$F^{(\rho)}_{i+1/2} \equiv \frac{1}{P_{i+1/2}+\rho_{i+1/2}}\frac{P_{i+1}-P_i}{r_{i+1}-r_i} + \frac{1}{f_{i+1/2}}\frac{f_{i+1}-f_i}{r_{i+1}-r_i},$$
$$F^{(f)}_{i+1/2} \equiv \kappa_G r^2_{i+1/2}P_{i+1/2} - \frac{2r_{i+1}}{f_{i+1}g^2_{i+1}}\frac{f_{i+1}-f_i}{r_{i+1}-r_i} + 1 - g^{-2}_{i+1/2},$$
$$F^{(g)}_{i+1/2} \equiv \kappa_G r^2_{i+1/2}\rho_{i+1/2} - \frac{2r_{i+1/2}}{g^3_{i+1/2}}\frac{g_{i+1}-g_i}{r_{i+1}-r_i} - 1 + g^{-2}_{i+1/2}.$$

Boundary Condition:

$$\rho_2 = 0, \quad g_0 = 1, \quad f_2 = \sqrt{1 - \frac{2GM_{tot}}{c^2 r_2}},$$
$$g_2 = f_2^{-1}(\text{or } \rho_0 = \rho_c)$$

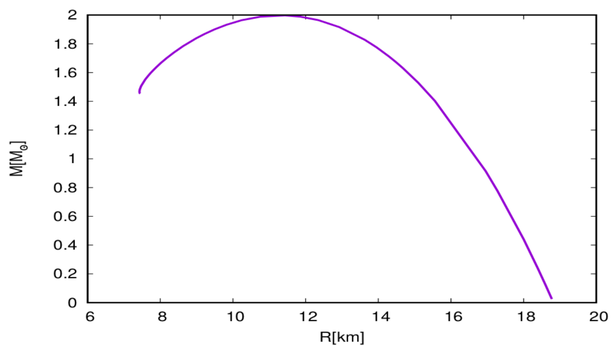


Solution for N-mesh star

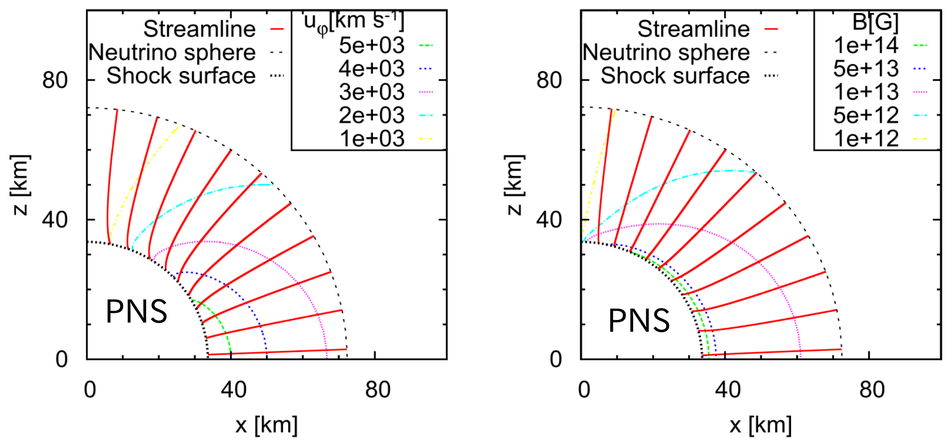


- Polytropic EOS($\Gamma = 2$)
- Solution depends on the resolution but converges.

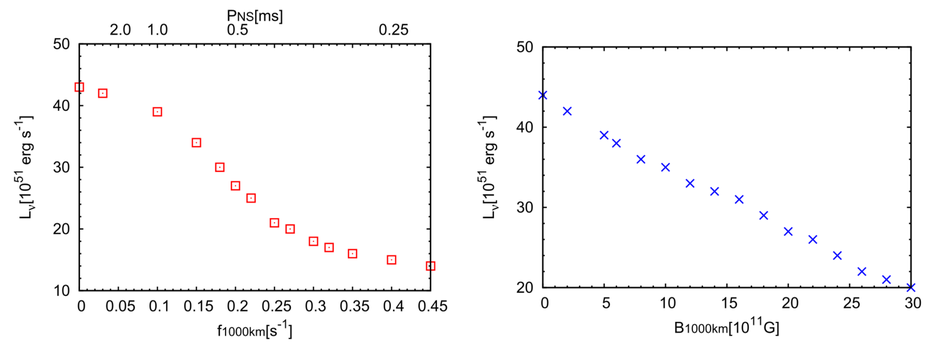
Mass-Radius Relation



Effects of Rotation and Magnetic Field on the Revival of a Stalled Shock in Supernova Explosions [2]



Streamlines in the meridian section (solid red lines) for the model with rotation (left) and toroidal magnetic field (right).



The critical luminosity as a function of the rotational rate (left) and the strength of the toroidal magnetic field (right)

References

[1] Press, W. H., Teukolsky, S. A., Vetterling, W. T., & Flannery, B. P. 1992, Numerical Recipes in FORTRAN. The Art of Scientific Computing
[2] Fujisawa, K., Okawa, H., et al, 2019, ApJ [3] Okawa, H., et al. (in prep)